Friday April 1

SECTION 7.2.
FEYNMAN DIAGRAMS IN MOMENTUM SPACE

First, let's consider an example:
ee scattering;
⇒ the Møller cross section.

The lowest order term is 2nd order in the interaction,

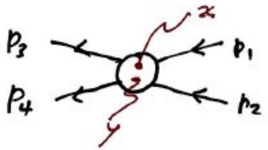
$$S^{(2)} = (1\,/2!)\,(ie)^2 \iint d^4x\, d^4y \;\langle f \,|\, T\; \bar{\psi}\gamma_\mu\psi\, A^\mu(x)\; \bar{\psi}\gamma_\nu\psi A^\nu(y) \,|\, i \rangle$$

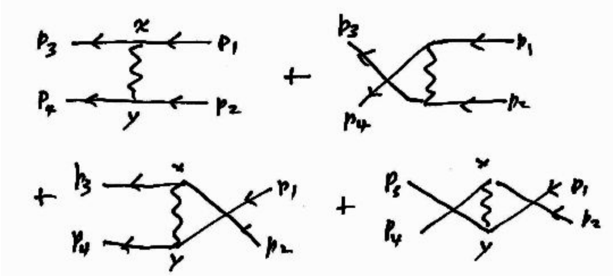
where $|\,i\,\rangle = |e_1\,;\,e_2\rangle$ and $|\,f\,\rangle = |e_3\,;\,e_4\,\rangle$

$S^{(2)} = (1/2!)\,(ie)^2 \iint d^4x\, d^4y$

$\langle f \,|\, T\ \bar{\psi}\gamma_\mu\psi\, A^\mu(x)\ \bar{\psi}\gamma_\nu\psi A^\nu(y) \,|\, i \rangle$

Apply Wick's theorem and the coordinate space Feynman rules.





- Wick's theorem gives us 4 terms.

However, $S_{22} = S_{11}$ and $S_{21} = S_{12}$ because we integrate over d^4x and d^4y.

(E.g., in S_{22} change the variables of integration from x,y to y',x' ; then drop the primes; the result is S_{11} .)

So, $S = (S_{11}+S_{12}) \times 2$;

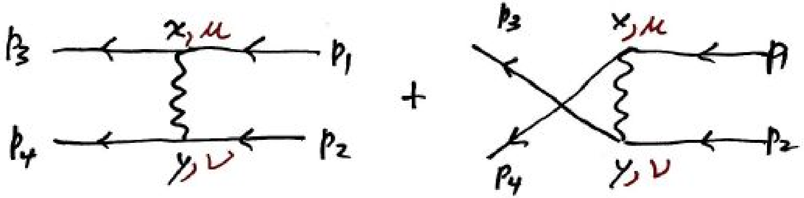
the 2 will cancel the 1 / 2!.

Note: The 1 / n! from the exponential series, will always cancel n! permutations of the vertex positions $\{ x_1 , x_2 , x_3 , \ldots , x_n \}$.

This leads to a Feynman rule:

(1) Draw all the *topologically distinct* diagrams with the specified external lines.

For ee scattering there are two Feynman diagrams,



The corresponding S-matrix elements are

$$S_t = \iint d^4x\, d^4y\ e^{ip_3\cdot x} e^{ip_4\cdot y} e^{-ip_1\cdot x} e^{-ip_2\cdot y} \prod_{a=1}^{4} \left(\frac{2m}{2E_a\Omega}\right)^{1/2} \bar{u}_3\, ie\gamma_\mu u_1\, \bar{u}_4\, ie\gamma_\nu u_2\, D_F^{\mu\nu}(x-y)$$

$$S_u = \iint d^4x\, d^4y\ e^{ip_3\cdot y} e^{ip_4\cdot x} e^{-ip_1\cdot x} e^{-ip_2\cdot y} \prod_{a=1}^{4} \left(\frac{2m}{2E_a\Omega}\right)^{1/2} \bar{u}_3\, ie\gamma_\nu u_2\, \bar{u}_4\, ie\gamma_\mu u_1\, D_F^{\mu\nu}(x-y)$$

Transform S to momentum space.

$$D^{\mu\nu}(\boldsymbol{x}-\boldsymbol{y}) = (2\pi)^{-4} \int d^4k \, (-g^{\mu\nu}/k^2) \, e^{\, i\, k.(x-y)}$$

For the case S_t : the integral d^4x gives

$$\int d^4x \; e^{ip_3 \cdot x} \, e^{-ip_1 \cdot x} \, e^{ik\cdot x} = (2\pi)^4 \, \delta^4(p_3 - p_1 + k)$$

$$\therefore \; k^\mu = p_1^\mu - p_3^\mu$$

Comment:
That would be for infinite spacetime.
Instead, we normalize plane waves in a finite volume Ω, so then the result should be

$$\Omega \, \delta_K(\vec{k}, \vec{p}_1 - \vec{p}_3) \, 2\pi \, \delta(E_3 - E_1 + k^0)$$

These are the same in the limit $\Omega \rightarrow \infty$,
but ...

The integral d^4y :

$$\int d^4y \; e^{ip_4 \cdot y} \, e^{-ip_2 \cdot y} \, e^{-ik\cdot y} = (2\pi)^4 \, \delta^4(p_4 - p_2 - k)$$

$$\therefore \; k^\mu = p_4^\mu - p_2^\mu$$

$$\therefore \; p_3^\mu + p_4^\mu = p_1^\mu + p_2^\mu$$

The transformation from coordinate space to momentum space gave us some delta functions for 4-momentum conservation. This leads to another Feynman rule:
(2) 4-momentum is conserved at every vertex.

$$S_t = \int \frac{d^4k}{(2\pi)^4} \, (2\pi)^8 \, \delta^4(k - p_1 + p_3) \, \delta^4(k + p_2 - p_4) \prod_{a=1}^{4} \left(\frac{2m}{2E_a\Omega}\right)^{1/2} \bar{u}_3 \, ie\gamma_\mu \, u_1 \, \bar{u}_4 \, ie\gamma_\nu \, u_2 \, \frac{-g^{\mu\nu}}{k^2}$$

$$= (2\pi)^4 \, \delta^4(P_f - P_i) \prod_{a=1}^{4} \left(\frac{2m}{2E_a\Omega}\right)^{1/2} \mathcal{M}_t$$

which defines the new matrix element $\mathcal{M}$.
(The other factors are the same in S_t and S_u

Result $\mathcal{M}_t = e^2(\bar{u}_3 \gamma_\mu u_1)(\bar{u}_4 \gamma^\mu u_2)/(p_1 - p_3)^2$

Similarly, exercise,

$$\mathcal{M}_u = -e^2(\bar{u}_3 \gamma_\mu u_2)(\bar{u}_4 \gamma^\mu u_1)/(p_1 - p_4)^2$$

Recall the Mandelstam variables,

$$s = (p_1+p_2)^2 = 2m^2 + 2p_1\cdot p_2 = (p_3+p_4)^2 = 2m^2 + 2p_3\cdot p_4$$

$$t = (p_1-p_3)^2 = 2m^2 - 2p_1\cdot p_3 = (p_4-p_2)^2 = 2m^2 - 2p_2\cdot p_4$$

$$u = (p_1-p_4)^2 = 2m^2 - 2p_1\cdot p_4 = (p_3-p_2)^2 = 2m^2 - 2p_2\cdot p_3$$

The Feynman rules in momentum space are rules for calculating the matrix element $\mathcal{M}$.

The S-matrix has some normalization factors that are not included in $\mathcal{M}$.
These normalization factors are

$$\prod_i (2\,\Omega\,E_i)^{-1/2} \quad \text{and} \quad \prod_f (2\,\Omega\,E_f)^{-1/2}$$

and $\prod_e (2m)^{1/2}$.

These factors are not part of $\mathcal{M}$.

The matrix element for ee (Møller) scattering

$$\mathcal{M}_t = e^2\ (\bar{u}_3\,\gamma_\mu\,u_1)\,(\bar{u}_4\,\gamma^\mu\,u_2)\ /t$$

$$\mathcal{M}_u = -e^2\ (\bar{u}_3\,\gamma_\mu\,u_2)\,(\bar{u}_4\,\gamma^\mu\,u_1)\ /u$$

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u$$

Note these other Feynman rules:
(3) A spinor for every external electron and positron.
(4) A polarization vector for every external photon.
(5) A propagator $S_F(p)$ for every (internal) electron line.
(6) A propagator $D_F^{\mu\nu}(q)$ for every (internal) photon line.
(7) A minus sign for exchanging 2 electrons.

To complete the calculation of the Møller cross section, we will need:

- $|\mathcal{M}|^2$;
- and the average over initial spins and sum over final spins, i.e.,

  $$\tfrac{1}{2} \sum_{\lambda_1} \tfrac{1}{2} \sum_{\lambda_2} \sum_{\lambda_3} \sum_{\lambda_4} [\ldots]$$

.

Let's go ahead and calculate that now.

Homework Problem X:
Plot the Moller cross section.

The cross section.

The transition probability is the square of the S-matrix element,

$$P = | S |^2 .$$

The transition rate is the probability per unit time,

$$w = P/(2T)$$

(evolution from -T to T)

The cross section is the rate divided by the incident flux, and the incident flux is

$$\Phi = \text{density} \times \text{velocity} = v_{rel} / \Omega ;$$

(1 particle in the volume Ω)

Thus $d\sigma = w / \Phi =$ $$\boxed{\frac{|S|^2 \Omega}{2T v_{rel}}}$$