Chapter 5. Photons: Covariant Theory
5.1. The classical fields ✔
5.2. Covariant quantization ✔
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Chapter 6. The S-Matrix Expansion
6.1. Natural Dimensions and Units ✔
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Chapter 7. Feynman Diagrams and Rules in QED
7.1. Feynman diagrams in configuration space ✔
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7.3. Feynman rules for QED ✔
7.4. Leptons (SKIP THIS; NEUTRINOS)

Chapter 8. QED Processes in Lowest Order
8.1. The cross section
8.2. Spin sums
8.3. Photon polarization sums
8.4-7. Examples
8.8-9. Bremsstrahlung

SECTION 7.3. FEYNMAN RULES FOR QED

The Feynman rules for QED are summarized in Appendix B.

(1) Draw all topologically distinct diagrams with the specified initial and final particles;
(2) 4-momentum is conserved at every vertex;
(3) a factor $i\, e\, \gamma^{\mu}$ at every vertex;
(4) spinor factors for incoming and outgoing electrons and positrons;
(5) polarization factors for incoming and outgoing photons;
(6) propagators for internal lines;
(7) sign change for fermion exchanges.

The result is the matrix element $\mathcal{M}_{fi}$.

(8) $$S_{fi} = \delta_{fi} + (2\pi)^4\, \delta^4(P_f - P_i)\, \left(\prod_{i,f} N\right)\, \mathcal{M}_{fi}\ .$$

To understand ...

Eq (8.1)

This defines $\mathcal{M}$

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4(P_f - P_i) \prod_i \frac{1}{\sqrt{2\Omega E_i}} \prod_f \frac{1}{\sqrt{2\Omega E_f}} \prod_\ell (2m_\ell)^{1/2} \mathcal{M}$$

This is the formula for the cross section

Eq. (8.8)

$$d\sigma = (2\pi)^4 \delta^4(P_f - P_i) \left(4E_1E_2 v_{rel}\right)^{-1} \prod_\ell (2m_\ell) \prod_f \frac{d^3p_f}{(2\pi)^3 2E_f} |\mathcal{M}|^2$$

SECTION 8.1. THE CROSS SECTION

The transition probability is the square of the S-matrix element,

$$P = | S |^2 .$$

The transition rate is the probability per unit time,

$$w = P/(2T) .$$

(time evolution from −T to T)

The cross section is the rate divided by the incident flux; and the incident flux is

$$\Phi = \text{density} \times \text{velocity} = v_{rel} / \Omega \; ;$$

(1 incident particle in the volume Ω)

Thus $$d\sigma = w / \Phi = \boxed{\frac{|S|^2 \, \Omega}{2T \, v_{rel}}} .$$

Now, the S-matrix (for finite volume Ω, and time evolution from t = − T to + T) is

$$S_{fi} = \text{“ } (2\pi)^4 \, \delta^4(P_f - P_i) \; (\prod N) \; \mathcal{M}_{fi} \text{ ”}$$

$$S_{fi} = \Omega \, \delta_K(\mathbf{P}_f , \mathbf{P}_i) \int_{-T}^{T} \exp\{i(E_f - E_i)t\} dt \; (\prod N) \; \mathcal{M}_{fi}$$

$$|S_{fi}|^2 = \Omega^2 \, \delta_K(\mathbf{P}_f , \mathbf{P}_i) \; (\prod N^2) \; |\mathcal{M}_{fi}|^2 \; \{2 \sin(T\Delta)/\Delta\}^2$$

where $\Delta = E_f - E_i$.

So $$d\sigma = \Omega^3 \, \delta_K(\mathbf{P}_f , \mathbf{P}_i) \, (\prod N^2) \, |\mathcal{M}_{fi}|^2 / v_{rel} \; \{2 \sin(T\Delta)/\Delta\}^2 / (2T)$$

So, $d\sigma = \Omega^3\, \delta_K(\mathbf{P}_f, \mathbf{P}_i)\, (\prod N^2)\, |\mathcal{M}_{fi}|^2 / v_{rel}$
$\{2 \sin(T\Delta)/\Delta\}^2 /(2T)$

Recall $N = \sqrt{(1/(2E\Omega))}$ for photons;
and $N = \sqrt{(2m/(2E\Omega))}$ for electrons.

Also $\{2 \sin(T\Delta)/\Delta\}^2 /(2T)$
$\to (2\pi)\, \delta(\Delta)$ as $T \to \infty$.

Thus

$d\sigma = \Omega^{3-ne}\, \delta_K(\mathbf{P}_f, \mathbf{P}_i)\, (2\pi)\, \delta(\Delta)\, |\mathcal{M}_{fi}|^2 / v_{rel}$
$\prod_e (1/2E_e) \prod_l (2m_l)$

$$F_T(\Delta) = \int_{-T}^{T} e^{it\Delta}\, dt = \frac{2 \sin(T\Delta)}{\Delta}$$

$$\lim_{T\to\infty} F_T(\Delta) = 2\pi\, \delta(\Delta)$$

$$\lim_{T\to\infty} \frac{1}{T} F_T^2(\Delta) = 4\pi\, \delta(\Delta)$$

<u>The total cross section:</u>
sum over final states

$$\sigma = \sum_f d\sigma = \prod_f \sum_{\vec{p}_f} d\sigma$$

$$= \prod_f \int \frac{\Omega\, d^3p_f}{(2\pi)^3}\, d\sigma$$

$$= \Omega^{2-n_i}\, (2\pi)^4\, \delta(P_f - P_i)\, |\mathcal{M}_{fi}|^2 / v_{rel}$$

$$\prod_f \int \frac{d^3p_f}{(2\pi)^3\, 2E_f} \prod_i \frac{1}{2E_i} \prod_e (2m_e)$$

v_{rel} = relative velocity of projectile and target

• In the lab frame

$\vec{p}_1, m_1 \longrightarrow \qquad \vec{p}_2 = 0, m_2$

Recall from special relativity,

$\vec{p} = \gamma m \vec{v}$ and $E = \gamma m c^2$

$$\therefore \quad v_{rel} = \frac{|\vec{p}_1|}{E_1}$$

• In the Center of Mass frame

$\vec{p}_1, m_1 \longrightarrow \qquad \times \qquad \longleftarrow -\vec{p}_1, m_2$

$(\vec{p}_1 + \vec{p}_2 = 0)$

$$v_{rel} = |\vec{p}_1| \left(\frac{1}{E_1} + \frac{1}{E_2} \right) \quad \text{or} \quad E_1 E_2 v_{rel} = |\vec{p}_1| (E_1 + E_2)$$

• In an arbitrary Lorentz frame ($\vec{p}_1$ and $\vec{p}$ being collinear)

$$E_1 E_2 v_{rel} = \left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{1/2}$$

Check: in the lab frame,

$(\vec{p}_2 = 0, \; E_2 = m_2)$

$$\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right]^{1/2} = \left[(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2)^2 - m_1^2 m_2^2 \right]^{1/2}$$

$$= \left[(E_1 m_2)^2 - m_1^2 m_2^2 \right]^{1/2} = |\vec{p}_1| m_2$$

and $E_1 E_2 v_{rel} = E_1 m_2 \frac{|\vec{p}_1|}{E_1} = |\vec{p}_1| m_2$ ✓

Recall the relativistic addition of velocities,

$$v' = \frac{v + u}{1 + vu/c^2}.$$

The center of mass cross section

... for the process $p_1 + p_2 \to p_3 + p_4$.

The total cross section is

$$\sigma = (2\pi)^4 \, \delta^4(P_f - P_i) \, (4E_1E_2 v_{rel})^{-1} \prod_\ell (2m_\ell) \prod_f \int \frac{d^3p_f}{(2\pi)^3 2E_f} \, |\mathcal{M}_{fi}|^2$$

We have some integrals to do,

... d^3p_3 and d^3p_4

§ Use the 3-momentum delta function,

$\delta^3(\mathbf{p}_3+\mathbf{p}_4-\mathbf{p}_1-\mathbf{p}_2)$

to do the integral over $\mathbf{p}_4$.

§ $d^3p_3 = p_3^{\,2}\, dp_3\, d\Omega_3$

§ Use the energy delta function,

$\delta(E_3+E_4-E_1-E_2)$

to do the integral over p_3 .

§ Divide by $d\Omega_3$ to get the differential cross section.

$$\frac{d\sigma}{d\Omega_3} = (2\pi)^{-2} \int p_3^2\, dp_3\, \delta(E_3+E_4-E_1-E_2) \, (4E_1E_2\, 4E_3E_4\, v_{rel})^{-1} \prod_\ell (2m_\ell) |\mathcal{M}_{fi}|^2$$

$$\frac{d\sigma}{d\Omega_3} = (2\pi)^{-2} \int p_3^2 dp_3 \;\; \delta(E_3+E_4-E_1-E_2)$$

$$\left(4E_1E_2 \, 4E_3E_4 \, v_{rel}\right)^{-1} \prod_\ell (2m_\ell) |\mathcal{M}_{fi}|^2$$

$$= \left(64\pi^2 E_3E_4 E_1E_2 v_{rel}\right)^{-1} \frac{p_3^2}{\left|\frac{\partial}{\partial p_3}(E_3+E_4)\right|}$$

$$\prod_\ell (2m_\ell) |\mathcal{M}_{fi}|^2$$

In the CoM frame,

$E_3 = \sqrt{p_3^2+m_3^2}$ and $E_4 = \sqrt{p_3^2+m_4^2}$

$\partial(E_3+E_4)/\partial p_3 = \dfrac{p_3}{E_3} + \dfrac{p_3}{E_4} = |\vec{p}_3| \dfrac{E_1+E_2}{E_3E_4}$

and $E_1E_2 v_{rel} = |\vec{p}_1| (E_1+E_2)$

$$\left(\frac{d\sigma}{d\Omega_3}\right)_{CoM.} = \frac{1}{64\pi^2 (E_1+E_2)^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \prod_\ell (2m_\ell) |\mathcal{M}_{fi}|^2$$

(eq 8.18)

Example

The Møller cross section

⇒ $e_1 + e_2 \rightarrow e_3 + e_4$

⇒ $m_1 = m_2 = m_3 = m_4 = m_e$

⇒ $|\mathbf{p}_1| = |\mathbf{p}_3|$

⇒ $(E_1 + E_2)^2 = s$ (Mandelstam variable)

$$\left(\frac{d\sigma}{d\Omega_3}\right)_{CoM} = \frac{1}{64\pi^2 s} 4e^4 \left\{ \frac{1}{t^2}\left[\frac{1}{2}(s-2m^2)^2 + \frac{1}{2}(2m^2-u)^2 + 2m^2 t\right] + \frac{1}{u^2}\left[\frac{1}{2}(s-2m^2)^2 + \frac{1}{2}(2m^2-t)^2 + 2m^2 u\right] + \frac{1}{tu}(s-2m^2)(s-6m^2) \right\}$$

The homework problem is to plot graphs of $(d\sigma/d\Omega_3)_{CoM}$ versus θ_3, for $\sqrt{(s)} = 3.16$, 10 and 31.6 MeV.

Plot the cross sections in *millibarns*.

Another example

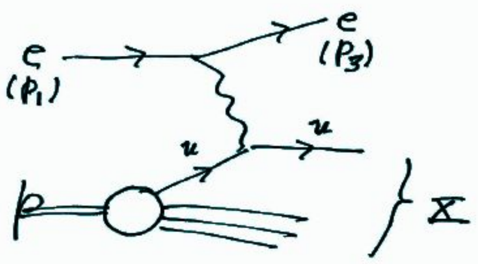
Deep inelastic electron scattering

These experiments started ~1970 at SLAC. The process is e + P → e + X, where P is a proton and X is *any set* of particles.

The experiment observes the final electron, and measures the 3-momentum, $\mathbf{p}_3$.

This experiment showed that the proton consists of "partons".

Later experiments on the same process showed that partons are "quarks" and "gluons".



$$d\sigma(ep) = \int_0^1 d\sigma(eu)\, f_u(x)\, dx$$

The homework problem is to calculate the differential cross section dσ/dt for electron-quark scattering; i.e., for the process e + u-quark → e + u-quark. (similar but simpler than ee scattering)