An example of the weak interaction - the muon decay rate

Page 373 in Mandl and Shaw.

The weak interaction history:

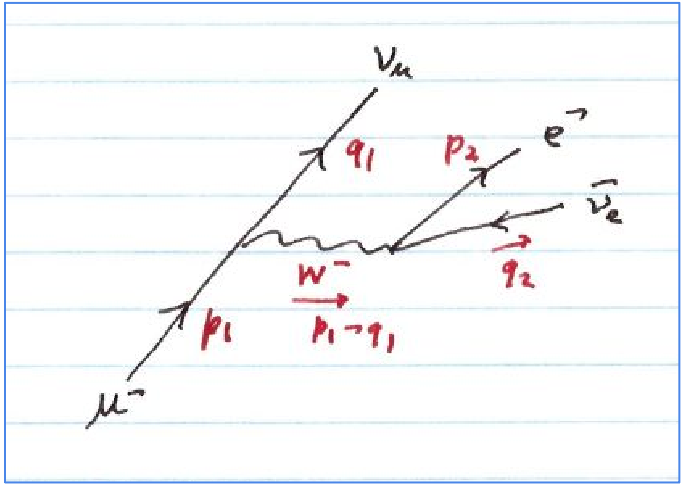
= Fermi's point coupling theory (30's)
= V−A parity violation (50's)
= The intermediate vector boson theory (50's)
= The Weinberg Salam model (60's)
= Discovery of W and Z bosons (80's)

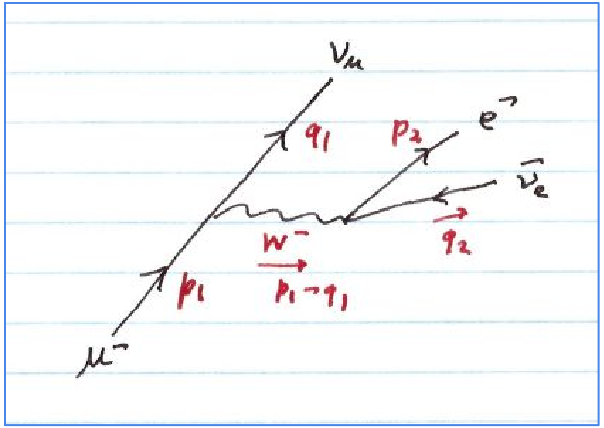
The mean lifetime of a muon in its rest frame is 2.2 μs. We'll calculate that from the Fermi weak coupling. The decay is

$$\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e$$

$$p_1^\mu = q_1^\mu + p_2^\mu + q_2^\mu$$

The Feynman diagram





$$\mathcal{M} = \bar{u}(q_1)\, g \gamma^\mu (1-\gamma_5)\, u(p_1)$$
$$\bar{u}(p_2)\, g \gamma^\nu (1-\gamma_5)\, v(q_2)$$
$$D_{\mu\nu}(p_1 - q_1)$$

The W-boson propagator is

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu} - k_\mu k_\nu / M_W^2}{k^2 - M_W^2}$$

But M_W = 80 GeV is much larger than the muon mass, so we can approximate

$$D_{\mu\nu}(k) = -g_{\mu\nu}/M_W^2$$

Also, we define the Fermi weak coupling G_F by

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{M_W^2}$$

$$\mathcal{M} = \frac{G}{\sqrt{2}} \; \bar{u}(q_1)\, \gamma^\mu (1-\gamma_5)\, u(p_1)$$
$$\bar{u}(p_2)\, \gamma_\mu\, (1-\gamma_5)\, v(q_2)$$

Calculate the square of *M*.

For example,

$$\bar{u}(q_1)\, \gamma^\mu\, (1-\gamma_5)\, u(p_1) \left[\bar{u}(q_1) \gamma^\nu (1-\gamma_5)\, u(p_1) \right]^\dagger$$

$$= \bar{u}(q_1)\, \gamma^\mu (1-\gamma_5) u(p_1)\, u^\dagger(p_1) \underbrace{(1-\gamma_5^\dagger)(\gamma^\nu)^\dagger (\gamma^0)^\dagger}\, u(q_1)$$

$$= \gamma^0 \gamma^0 (1-\gamma_5) (\gamma^\nu)^\dagger \gamma^0$$

$$= \gamma^0 \; \underbrace{\gamma^0 (1-\gamma_5)\, \gamma^0}_{1+\gamma_5} \; \underbrace{\gamma^0 (\gamma^\nu)^\dagger \gamma^0}_{\gamma^\nu}$$

$$= \gamma^0\, \gamma^\nu\, (1-\gamma_5)$$

$$= \bar{u}(q_1)\, \gamma^\mu\, (1-\gamma_5)\, u(p_1)\, \bar{u}(p_1)\, \gamma^\nu (1-\gamma_5)\, u(q_1)$$

$$= \mathrm{Tr}\left\{ \gamma^\mu (1-\gamma_5)\; u\bar{u}(p_1)\; \gamma^\nu\, (1-\gamma_5)\; u\bar{u}(q_1) \right\}$$

For an unpolarized muon, average over initial spin and sum over final spins

$$\frac{1}{2}\sum_{\lambda_1} u\bar{u}(p_1) = \frac{1}{4M}(\not{p}_1 + M)$$

$$\sum_{\lambda} u\bar{u}(q_1) = \frac{\not{q}_1}{2m_\nu} \quad (\nu_\mu \text{ mass} = 0)$$

$$\sum_{\lambda} u\bar{u}(p_2) = \frac{1}{2m}(\not{p}_2 + m)$$

$$\sum_{\lambda} v\bar{v}(q_2) = \frac{\not{q}_2}{2m_\nu} \quad (\nu_e \text{ mass} = 0)$$

Result for unpolarized decay

$$|\mathcal{M}|^2 = \frac{G^2}{2}\,\frac{1}{2}\,\mathrm{Tr}\,\gamma^\mu(1-\gamma_5)(\not{p}_1+M)\gamma^\nu(1-\gamma_5)\not{q}_1$$

$$\mathrm{Tr}\,\gamma_\mu(1-\gamma_5)\not{q}_2\,\gamma_\nu(1-\gamma_5)(\not{p}_2+m)$$

$$\prod_{\ell}(1/2m_\ell)$$

Muon tensor (p_1^μ and q_1^μ)

$$M^{\mu\nu} = \mathrm{Tr}\,\gamma^\mu(1-\gamma_5)(\not{p}_1+M)\gamma^\nu(1-\gamma_5)\not{q}_1$$

$\mathrm{Tr}\,\gamma\gamma\gamma = 0$

$$= \mathrm{Tr}\,\gamma^\mu(1-\gamma_5)\,\underbrace{\not{p}_1\,\gamma^\nu(1-\gamma_5)\not{q}_1}_{(1-\gamma_5)\not{p}_1\gamma^\nu\not{q}_1}$$

$$\underbrace{(1-\gamma_5)(1-\gamma_5)}_{1+1-2\gamma_5 \,=\, 2(1-\gamma_5)}$$

$$= 2\,\mathrm{Tr}\,(1+\gamma_5)\gamma^\mu\not{p}_1\gamma^\nu\not{q}_1$$

$$= 2(S^{\mu\nu} + A^{\mu\nu})$$

Symmetric part

$$S^{\mu\nu} = 4\left[\, p_1^\mu q_1^\nu + q_1^\mu p_1^\nu - g^{\mu\nu}\, p_1\cdot q_1 \right]$$

Antisymmetric part

$$A^{\mu\nu} = \mathrm{Tr}\ \gamma_5^{i}\ \gamma^\mu \not{p}_1 \gamma^\nu \not{q}_1$$

$$= p_{1\alpha}\, q_{1\beta}\ \mathrm{Tr}\ \gamma_5\, \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta$$

$$= p_{1\alpha}\, q_{1\beta}\ 4i\, \epsilon^{\mu\alpha\nu\beta}$$

$$M^{\mu\nu} = 2\left[\, S^{\mu\nu}(p_1, q_1) + A^{\mu\nu}(p_1, q_1) \right]$$

Work out the electron tensor similarly; then the result is

$$\overline{|\mathcal{M}|^2} = \frac{G^4}{4} M^{\mu\nu} E_{\mu\nu}$$

$$= G^4 \left[\, S_M^{\mu\nu}(p_1 q_1) + A_M^{\mu\nu}(p_1 q_1) \right] \left[\, S_{\mu\nu\,m}(q_2 p_2) + A_{\mu\nu\,m}(q_2 p_2) \right]$$

We'll have S.S and A.A; but S.A is 0.

<u>Exercise</u>

$$S_M^{\mu\nu} S_{\mu\nu\,m} = 32 \left\{ p_1\cdot p_2\ q_1\cdot q_2 + p_1\cdot q_2\ p_2\cdot q_1 \right\}$$

$$A_M^{\mu\nu} A_{\mu\nu\,m} = 32 \left\{ p_1\cdot q_2\ p_2\cdot q_1 - p_1\cdot p_2\ q_1\cdot q_2 \right\}$$

The SIGN is <u>critical</u>!

Result

$$\overline{|M|^2} = 64\ G^2\ p_1 \cdot q_2\ \ p_2 \cdot q_1\ \ / \,\Pi(\,2m_l\,)$$

Lifetime. (page 464 ; equation 16.36)

$$d\Gamma = (2\pi)^4\, \delta^4(P_f - P_i)\, \frac{1}{2E_i} \left[\prod_\ell (2m_\ell)\right] \prod_f \left(\frac{d^3p_f}{(2\pi)^3\, 2E_f}\right) |M|^2$$

$d\Gamma$ = the differential decay rate

$$\text{Lifetime} = \tau = \sum_{\text{branches}} \frac{\text{branching ratio}}{\Gamma}$$

The neutrinos are unobservable.
So, we'll integrate over $\mathbf{q_1}$ and $\mathbf{q_2}$ and get the differential rate w.r.t. electron momentum.

<u>Phase space integral for the two neutrinos</u>

The 3-body final state, and the muon rest frame ; neglect m_e and m_{ν_μ} ;

$$\frac{1}{2M} \int \frac{d^3p_2}{(2\pi)^3\, 2E_{p_2}} \int \frac{d^3q_1}{(2\pi)^3\, 2q_1} \int \frac{d^3q_2}{(2\pi)^3\, 2q_2}\, |M|^2 \left(\prod_\ell (2m_\ell)\right)$$

$$(2\pi)^4\, \delta(M - \underset{\uparrow}{E_e} - q_1 - q_2)\, \delta^3(\vec{p}_2 + \vec{q}_1 + \vec{q}_2)$$

$$E_e = \sqrt{p_2^2 + m_e^2}$$

Let $q^\mu = p_1^\mu - p_2^\mu = q_1^\mu + q_2^\mu$

In Rest frame : $q^0 = M - E_e$

$$\vec{q} = \vec{q}_1 + \vec{q}_2 = -\vec{p}_2$$

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$$d\Gamma = \frac{1}{2M} \frac{d^3 p_2}{(2\pi)^3 2E_e} \frac{64 G^2 p_{1\alpha} p_{2\beta}}{(2\pi)^2 \, 2 \cdot 2}$$

$$\times \int \frac{d^3 q_1}{q_1} \int \frac{d^3 q_2}{q_2} \delta^4(q_1 + q_2 - q) q_2^{\alpha} q_1^{\beta}$$

Call this $I^{\alpha\beta}(q)$;
note that $I^{\alpha\beta}$ is symmetric in $\alpha\beta$.

Now use some tricks to calculate $I^{a\beta}(q)$.

$$I^{\alpha\beta}(q) = \int \frac{d^3 q_1}{|\vec{q}_1|} \int \frac{d^3 q_2}{|\vec{q}_2|} q_1^{\alpha} q_2^{\beta} \delta^4(q_1 + q_2 - q)$$

$I^{\alpha\beta}(q)$ is a Lorentz tensor (L.I.P.S.); and it only depends on the 4-vector q^{μ} . Therefore $I^{\alpha\beta}(q)$ must have the form

$$I^{\alpha\beta}(q) = g^{\alpha\beta} A(q^2) + q^{\alpha} q^{\beta} B(q^2)$$

where A and B are scalars.

① Consider $g_{\alpha\beta} I^{\alpha\beta} = 4A + q^2 B$

$$= \int \frac{d^3q_1}{q_1} \int \frac{d^3q_2}{q_2} \underbrace{q_1 \cdot q_2}_{} \delta^4(q_1 + q_2 - q)$$

$= \frac{1}{2}(q_1+q_2)^2 = \frac{1}{2}q^2$

because $q_1^2 = q_2^2 = 0$

$$= \frac{1}{2} q^2 I(q^2)$$

where $I(q^2) = \int \frac{d^3q_1}{q_1} \int \frac{d^3q_2}{q_2} \delta^4(q_1+q_2-q)$

I is a scalar, so we may evaluate it in any Lorentz frame.

Use the frame where $\vec{q} = 0$.

Then $q^2 = (q^0)^2$ and

$$I = \int \frac{d^3q_1}{|\vec{q}_1|} \frac{1}{|-\vec{q}_1|} \delta(|\vec{q}_1| + |-\vec{q}_1| - q^0)$$

$$= 4\pi \int \frac{q_1^2\, dq_1}{q_1^2} \delta(2q_1 - q^0) = 2\pi.$$

① $4A + q^2 B = \frac{1}{2} q^2 I = \pi q^2.$

② Now consider $q_\alpha q_\beta I^{\alpha\beta} = q^2 A + (q^2)^2 B$

$$= \int \frac{d^3q_1}{q_1} \int \frac{d^3q_2}{q_2} \; q\cdot q_1 \; q\cdot q_2 \; \delta^4(q_1+q_2-q)$$

$q\cdot q_1 = (q_1+q_2)\cdot q_1 = q_1\cdot q_2$ b/c $q_1^2=0$

$q\cdot q_2 = (q_1+q_2)\cdot q_2 = q_1\cdot q_2$ b/c $q_2^2=0$

$\therefore \; q\cdot q_1 \; q\cdot q_2 = (q_1\cdot q_2)^2 = (q^2/2)^2$

So $q^2 A + (q^2)^2 B = \frac{1}{4}(q^2)^2 I = \frac{\pi}{2}(q^2)^2$ ②

$A + q^2 B = \frac{\pi}{2} q^2$ ②

$4A + q^2 B = \pi q^2$ ①

$A = \frac{\pi}{6} q^2$ and $B = \pi/3$

$$I^{\alpha\beta}(q) = \frac{\pi}{6}\left\{ g^{\alpha\beta} q^2 + 2 q^\alpha q^\beta \right\}$$

The decay rate:

$$d\Gamma = \frac{1}{2M}\int \frac{d^3p_2}{(2\pi)^3 2E_e} \frac{4G^2}{\pi^2} \; p_{1\alpha} p_{2\beta} I^{\alpha\beta}$$

$$p_{1\alpha} p_{2\beta} I^{\alpha\beta} = \frac{\pi}{6}\left\{ p_1\cdot p_2 \, q^2 + 2\, p_1\cdot q \; p_2\cdot q \right\}$$

where $q^\mu = p_1^\mu - p_2^\mu$

= after a bit of algebra

$$= \frac{\pi}{6}\left\{ 3\, p_1\cdot p_2 \,(M^2+m^2) - 4(p_1\cdot p_2)^2 - 2M^2m^2 \right\}$$

or, in the muon rest frame

($\vec{p}_1 = 0$; $p_1\cdot p_2 = M E_e$)

$$= \frac{\pi}{6} M \left\{ 3E_e (M^2+m^2) - 4M E_e^2 - 2Mm^2 \right\}$$

Collect all the factors.

Note $\int d^3p_2 = \int p_2^{\,2}\, dp_2 \; d\Omega = 4\pi\, p_2\, E_2\, dE_2$.

So the rate, differential in electron energy, is

$$\frac{d\Gamma}{dE_e} = \frac{1}{2M}\,\frac{4\pi\, p_2 E_e}{(2\pi)^3\, 2E_e}\,\frac{4G^2}{\pi^2}\,\frac{\pi M}{6}\left\{3E_e\,(M^2+m^2) - 4ME_e^2 - 2Mm^2\right\}$$

$$\frac{d\Gamma}{dE_e} = \frac{G^2}{12\pi^3}\sqrt{E_e^2-m^2}\left\{3E_e\,(M^2+m^2) - 4ME_e^2 - 2Mm^2\right\}$$

(Eqn 16.66)

Muon decay rate

```
Remove["Global`*"]
GF = 1.16637*^-5 *UGeV^-2;
(* using units with hb = 1 and c = 1 *)
In[367]:= δΓ = GF^2/(12*Pi^3)*Sqrt[Ee^2 - me^2]*
     (3*Ee*(M^2 + me^2) - 4*M*Ee^2 - 2*M*me^2);
masses = {me → 0.511*^-3*UGeV, M → 0.106*UGeV};
test = δΓ /. Join[masses, {Ee → 0.033*UGeV}];
PowerExpand[test]
Out[370]= 7.84904 × 10^-18
(* restore units: *)
In[371]:= hb = (0.197*UGeV*Ufm)/(2.998*^23*Ufm/Us);
diffrate = δΓ/hb /. masses;
diffrate = diffrate /. {UGeV → 1, Us → 1};
Plot[diffrate, {Ee, 0, 0.053},
 AxesLabel → {"Ee [GeV]", "dR/dEe [GeV^-1 s^-1]"}]
Out[374]=
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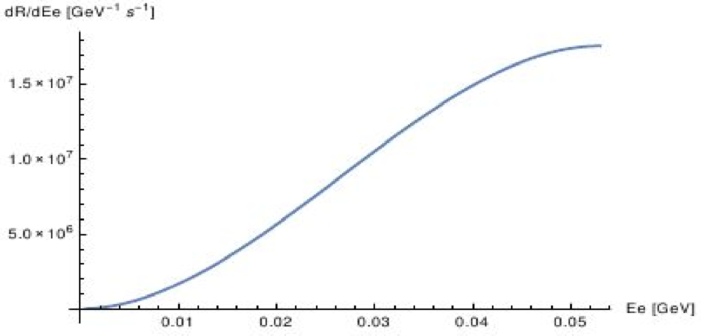

$$\frac{d\Gamma}{dE_e} = \frac{G^2}{12\pi^3}\sqrt{E_e^2-m^2}\,\Big\{3E_e(M^2+m^2) - 4ME_e^2 - 2Mm^2\Big\}$$

(Eqn 16.66)

Homework Problem.

(A) From the muon lifetime, 2.2 μs in the rest frame, calculate the Fermi weak coupling constant G_F .

(B) From the result, and the W mass, calculate the weak-mixing angle (Weinberg angle).

(C) According to Mandl and Shaw, the helicity of the electron from muon decay is approximately -1. Explain why.