Homework Assignment #6 Name______grading____________

due in class Friday, October 14

Cover Sheet : Staple this page in front of your solutions.

Write the *answers* (without calculations) on this page;
write the detailed *solutions* on your own paper.

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[27] Problem 3.16.*

Answer: The distance from the center of mass of the earth-sun system to the center of the sun is a fraction of the solar radius. The fraction is

$$450 / 7\text{x}10^5 = 6.4 \text{ x } 10^{-4}$$

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[28] Problem 3.20.** *(There is no answer to report here.)*

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[29] Problem 3.22.**

Answer: the coordinates (x,y,z) of the center of mass are

$$(0 , 0 , \tfrac{3}{8} R)$$

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[210] Problem 3.27.**

Answer: Let A = the area bounded by the orbit and T = the period of revolution. The relation between A and T is

$$A = (\ell / 2m) T$$

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[211] Problem 3.32.** *The moment of inertia is*

$$\tfrac{2}{5} M R^2$$

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[212] Problem 3.35.**

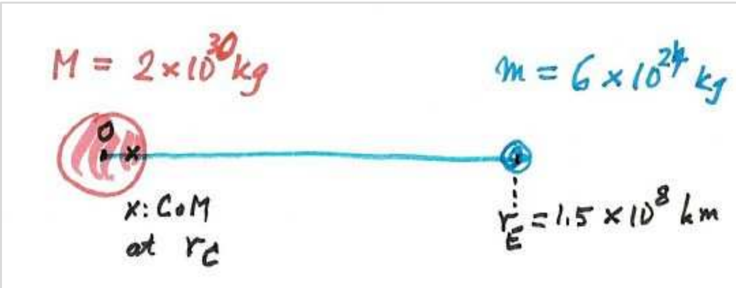
The acceleration of the center of mass of the disk is

$$\tfrac{2}{3} g \sin \gamma$$

Homework Assignment #6

Problem 3.16

Earth and Sun …



Calculate the position of the center of mass point.

Let r_C = the distance from the center of the sun to the CoM point.

$$(M+m)\, r_C = M\, r_S + m\, r_E$$

where $r_S = 0$ (taking the origin to be the center of the Sun)

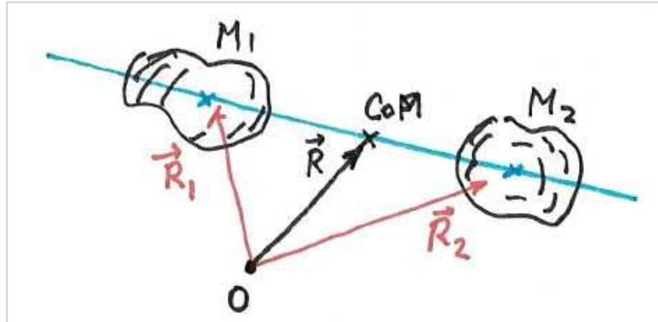
$$r_c = \frac{m\, r_E}{M+m} = \frac{6\times10^{24} \times 1.5\times10^{8}\ \text{km}}{2\times10^{30}}$$

$$r_c = 4.5\times10^{2}\ \text{km} = 450\ \text{km}$$

which is <<< the radius of the Sun ($R_S = 7 \times 10^5$ km).

Problem 3.20

CoM point of a system of two solid objects …



The center of mass position vector is

$$\mathbf{R} = 1/M \int \mathbf{r}\, dM$$

$$\frac{1}{M_1+M_2}\left\{\int_{B_1}\vec{r}\,dM+\int_{B_2}\vec{r}\,dM\right\}$$

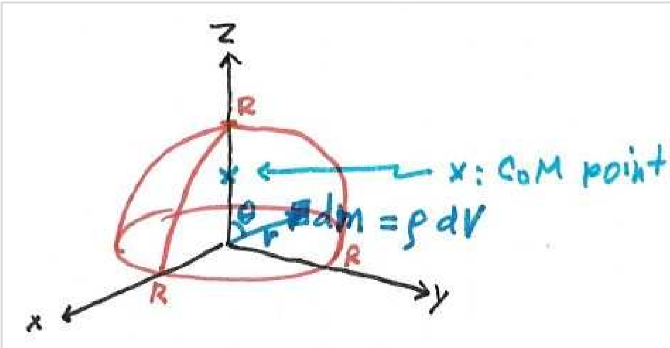
$$= \frac{1}{M_1+M_2}\left\{M_1\vec{R}_1+M_2\vec{R}_2\right\}$$

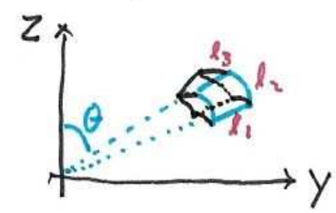
$$= \frac{M_1\vec{R}_1+M_2\vec{R}_2}{M_1+M_2}$$

which is the same as if the masses were concentrated at their center of mass points.

Problem 3.22

CoM of a uniform solid hemisphere ...





$\ell_1 = dr$

$\ell_2 = r\, d\theta$

$\ell_3 = r \sin\theta\, d\phi$

$dV = \ell_1 \ell_2 \ell_3 = r^2 dr \sin\theta\, d\theta\, d\phi$

By symmetry, the center of mass point is located on the z axis, at $z = z_C$.

$$z_C = 1/M \int z\, \rho\, dV$$

Using spherical polar coordinates, $dV = dr \times r\, d\theta \times r \sin\theta\, d\varphi$;

$dV = r^2\, dr\ \sin\theta\, d\theta\ d\varphi$ ← *always remember this!*

$$z_C = \frac{1}{M}\iiint r\cos\theta\ \rho\ r^2 dr \sin\theta\, d\theta\, d\phi$$
$$= \frac{\rho}{M}\int_0^R r^3 dr \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= \frac{\rho}{M}\ \frac{R^4}{4} \cdot \frac{1}{2} \cdot 2\pi = \frac{\pi}{4}\frac{\rho}{M} R^4$$

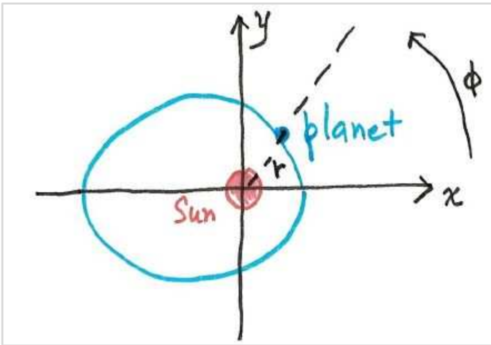
Mass density: $\rho = \dfrac{M}{V} = \dfrac{M}{\frac{2}{3}\pi R^3} = \dfrac{3M}{2\pi R^3}$

Thus $z_C = \frac{3}{8} R$.

Problem 3.27

Kepler's 2nd Law ...

A planet revolves around the fixed Sun; the orbit plane = xy



Position vector $\mathbf{r} = r\cos\varphi\ \boldsymbol{e}_x + r\sin\varphi\ \boldsymbol{e}_y = r\,\boldsymbol{e}_r$

Velocity vector $\boldsymbol{v} = d\mathbf{r}/dt$

$$= r'\,(\cos\varphi\ \boldsymbol{e}_x + \sin\varphi\ \boldsymbol{e}_y) + r\varphi'\,(-\sin\varphi\ \boldsymbol{e}_x + \cos\varphi\ \boldsymbol{e}_y)$$

$$= r'\,\boldsymbol{e}_r + r\,\varphi'\,\boldsymbol{e}_\varphi$$

(a) Angular momentum

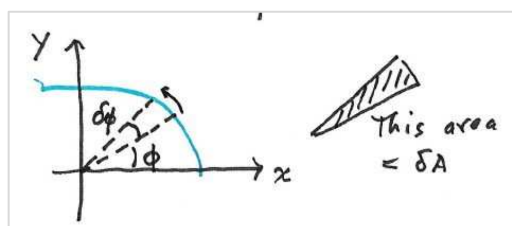
$$\boldsymbol{\ell} = \mathbf{r} \times m\,\boldsymbol{v} = mr\,\boldsymbol{e}_r \times (r'\,\boldsymbol{e}_r + r\,\varphi'\,\boldsymbol{e}_\varphi) = m\,r^2\,\varphi'\ \boldsymbol{e}_z$$

(b) The area swept out

during a small time δt is

$\delta A = ½ \times$ base $\times$ height

$$= ½\quad r\quad r\,\delta\varphi = ½\,r^2\ \delta\varphi$$



Thus the ***area rate*** is

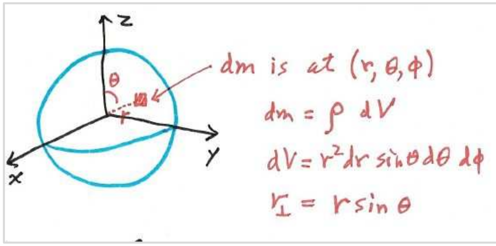
$$dA/dt = ½\,r^2\,\varphi' = \ell\ /(2m)$$ = *which is a constant*.

Kepler's second law:
The radial vector sweeps out equal areas in equal times.
To prove it, note that the area swept out during any time interval Δt is $\Delta A = \ell\ /(2m) \times \Delta t$, which only depends on the length of time.

Problem 3.32

Moment of inertia of a sphere …

Consider a uniform solid sphere, with mass M and radius R.



Calculate the moment of inertia about a diameter axis.

Divide the sphere into infinitesimal masses;

then $\qquad I = \int r_\perp^{\,2}\, dM$

Now $\qquad dM = \rho\, dV\,; \qquad dV = r^2\, dr\, \sin\theta\, d\theta\, d\varphi\,; \qquad r_\perp = r \sin\theta\,;$

so

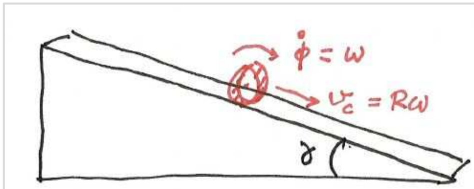
$$I = \int r_\perp^2\, dm = \int r^2 \sin^2\theta\, \rho\, r^2 dr\, \sin\theta\, d\theta\, d\phi$$
$$= \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} d\phi$$
$$= \rho\ \frac{R^5}{5} \cdot \frac{4}{3} \cdot 2\pi \ = \rho\, \frac{8\pi}{15} R^5$$

Mass density: $\qquad \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$

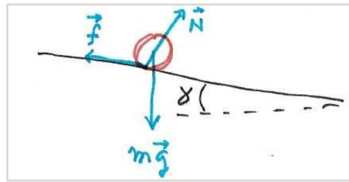
Thus $\qquad I = \frac{2}{5}\, M R^2$.

Problem 3.35 **6.6**

A disk rolling on an inclined plane …



(a) The free–body force diagram



(b) Derive the equation of motion, from the angular momentum and torque about an axis through the point P (= contact point).

$$\vec{L}\,(\text{about } P) = I_P\,\omega\,\hat{n}$$

$$I_P = I_c + MR^2 = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2$$

$$\left.\begin{aligned} \dot{L}_z &= \tfrac{3}{2}MR^2\,\dot{\omega} = \tfrac{3}{2}MR\,\dot{v} \\ \Gamma_z(\text{about } P) &= Mg\,R\sin\gamma \end{aligned}\right\}$$

$$\dot{v} = \frac{2}{3}\,g\sin\gamma$$

(c) Derive the equation of motion, from the angular momentum and torque about an axis through the point C (= center of the disk).

$$\vec{L} = I_c\,\omega\,\hat{n} = \tfrac{1}{2}MR^2\,\omega\,\hat{n}$$

$$\dot{L}_z = \tfrac{1}{2}MR^2\,\dot{\omega} = \tfrac{1}{2}MR\,\dot{v}$$

$$\Gamma_z(\text{about } o) = Rf$$

The center of mass motion:

$$M\dot{v} = Mg\sin\gamma - f$$

so $f = Mg\sin\gamma - M\dot{v}$

and $f = \dfrac{\Gamma}{R} = \tfrac{1}{2}M\dot{v}$

Thus $\dot{v} = \dfrac{2}{3}\,g\sin\gamma.$