Homework Assignment #7

Name______grading______________________

due in class Friday, October 21

Cover sheet : Staple this page in front of your solutions.

Write the *answers* (without calculations) on this page;
write the detailed *solutions* on your own paper.

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[31] Problem 4.3.** *Answer: Is **F** conservative? Why or why not?*

F is not conservative because W depends on the path.

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[32] Problem 4.8.** *Answer: The angle when the puck comes off the ..surface is* $\frac{2}{3}$ *arccos( ) = 0.841 = 48.2 degrees*

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[33] Problem 4.9.** *Answer: How does the frequency depend on g?*

The frequency = sqrt(k/m)/(2 π); the frequency does not depend on g.

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[34] Problem 4.10.*

Answer: Consider points on the line with x = 0 and z = 1;
what is the gradient of h? $\nabla h(0,y,1) = \{ ay, 0, 0 \} = a\, y\, \mathbf{e}_x$

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[35] Problem 4.18.** *Answer:*

$df = \nabla f \cdot d\mathbf{r} = |\nabla f|\,|d\mathbf{r}| \cos\theta$ is maximum at $\theta = 0$.

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[36] Problem 4.23.**

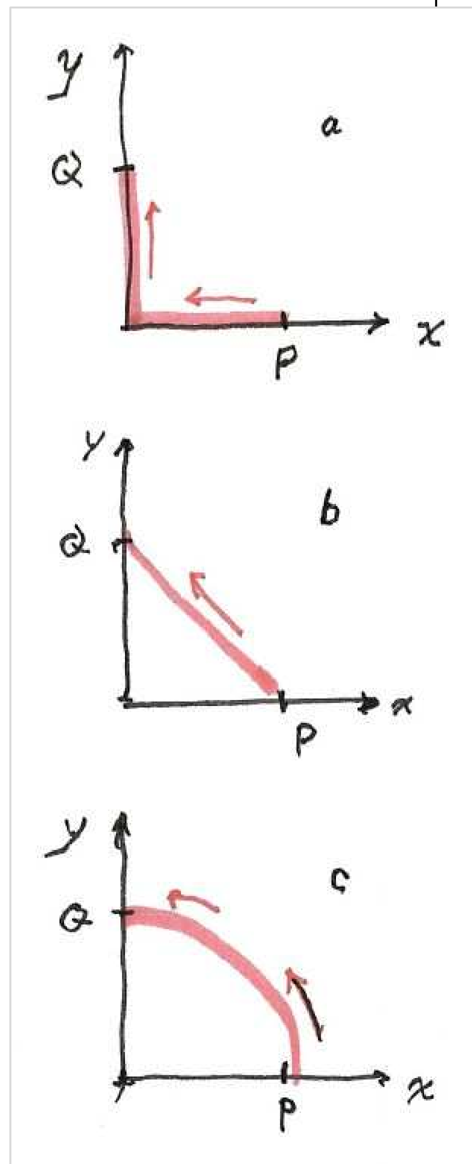
Answer: The potential energy function for part (b) is

U(x,y,z) = – k x y

Homework Assignment #7

Problem 4.3

$$\textbf{\textit{Work}} = \int \textbf{\textit{F . ds}} \ \ldots$$



Consider the force field $\mathbf{F(r)} = -\,y\, \boldsymbol{e_x} + x\, \boldsymbol{e_y}$.

Calculate the work done for 3 paths from P to Q.

(a) $W_a = \int_{\Gamma a} \mathbf{F\,.\,ds} = \int_{\Gamma a} \{ F_x\, dx + F_y\, dy \}$

$= \int_1^0 F_x(x,0)\, dx + \int_0^1 F_y(0,y)\, dy$

$= 0 + 0$ $\quad$ **= 0.**

(b) $W_b = \int_{\Gamma b} \mathbf{F\,.\,ds} = \int_{\Gamma b} \{ F_x\, dx + F_y\, dy \}$

Along Γ_b , $x + y = 1$; that is, $y = 1 - x$.

$\therefore \quad W_b = \int_1^0 \{ F_x\,(x,1-x)\, dx + F_y\,(x,1-x)(-dx) \}$

$= \int_1^0 \{ -(1-x) - x \}\, dx$ $\quad$ **= 1.**

(c) $W_b = \int_{\Gamma c} \mathbf{F\,.\,ds} = \int_{\Gamma c} \{ F_x\, dx + F_y\, dy \}$

Along Γ_c , $x^2 + y^2 = 1$; that is, $y = \sqrt{(1 - x^2)}$.

Continuing with Cartesian coordinates,

$\therefore \quad W_b = \int_1^0 \{ F_x\,(x, (1-x^2)^{1/2})$

$+ F_y\,(x, (1-x^2)^{1/2})\,(-x)\,(1-x^2)^{-1/2} \}\ dx$

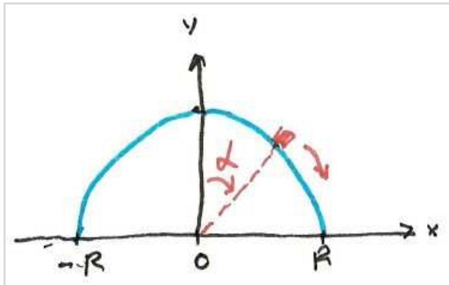
finish this

(It is easier to calculate W_c using polar coordinates.)

Answers: $\{ 0 , 1 , \pi/2 \}$

Problem 4.8

A puck slides down the surface of a fixed sphere …



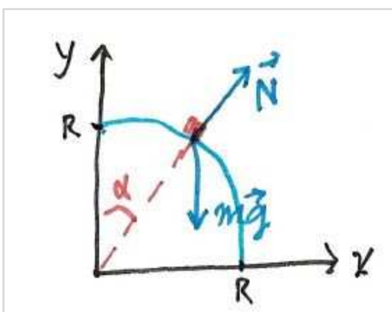
When the puck is a angle α, $y = R \cos \alpha$.

- Energy is conserved,

$$\tfrac{1}{2} m v^2 + mgy = \textit{constant} = \text{the initial value} = mgR \,;$$

Therefore, $v^2 = 2 g (R - y)$.

- Now calculate the normal force, when the puck is at angle α .

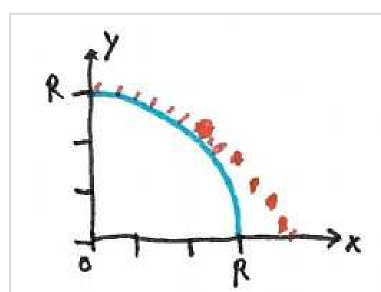


The radial acceleration is $a_r = - R \alpha'^2$;
the velocity is $v = R \alpha'$ (tangential) ;
thus $a_r = - v^2 /R$.

By Newton's second law, $a_r = F_r /m = (N - mg \cos \alpha) /m$.

So

- The puck comes off the surface when N = 0.



That is, $y = \tfrac{2}{3} R$.

Problem 4.9

Mass and spring suspended from the ceiling …

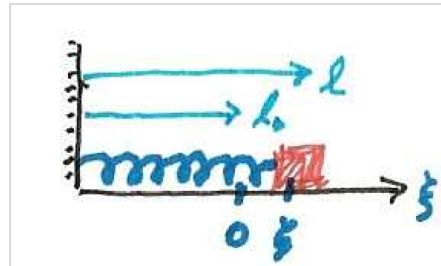
(a) First, consider the horizontal spring.

Unstressed length = ℓ_0 ;

force F = − k ($\ell - \ell_0$) = − k ξ ;

(ξ = displacement from horizontal equilibrium point)

potential energy = ½ k ξ^2 .



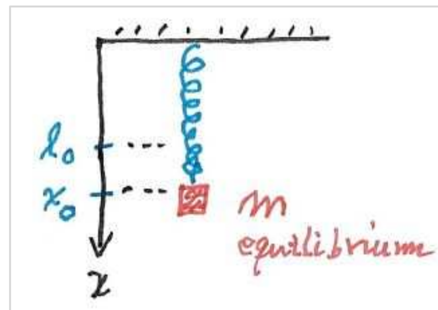
(b) Now, the spring is suspended from the ceiling.

The x axis points downward from the suspension point.

Let x_0 = the equilibrium position of m.

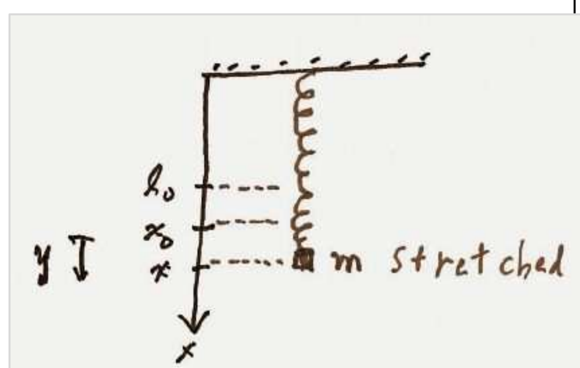
At equilibrium, $F_x = 0$;

the force is $F_x = mg - k(x - \ell_0)$, so $x_0 = \ell_0 + mg/k$.



(c) Now suppose the mass is displaced from the equilibrium point; let y = the displacement; i.e. $y = x - x_0$.

Calculate the total potential energy.



$$U = \frac{1}{2}k(x-\ell_0)^2 - mgx + \text{constant}$$

Reference point : $U(y=0) = 0$

$$= \frac{1}{2}k(x_0-\ell_0)^2 - mgx_0 + C$$

$$\therefore\ C = -\frac{1}{2}k(x_0-\ell_0)^2 + mgx_0$$

$$U(y) = \frac{1}{2}k(x_0+y-\ell_0)^2 - mg(x_0+y) - \frac{1}{2}k(x_0-\ell_0)^2 + mgx_0$$

$$U(y) = \frac{1}{2}k\,2(x_0-\ell_0)y + \frac{1}{2}ky^2 - mgy$$

$k(x_0-\ell_0) = mg$ by part (b).

$$U(y) = \frac{1}{2}ky^2 .$$

Problem 4.10

Examples of partial derivatives …

(a) $f(x,y,z) = a\,x^2 + b\,x\,y + c\,y^2$

$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
$= 2ax + by$	$= bx + 2cy$	$= 0$

(b) $g(x,y,z) = \sin(\,a\,xyz^2\,)$

$\partial g/\partial x$	$\partial g/\partial y$	$\partial g/\partial z$
$= ayz^2\cos(axyz^2)$	$= axz^2\cos(axyz^2)$	$= 2axyz\cos(axyz^2)$

(c) $h(x,y,z) = a\exp(\,xy/z^2\,)$; let $S = xy/z^2$

$\partial h/\partial x$	$\partial h/\partial y$	$\partial h/\partial z$
$= ay/z^2 \exp S$	$= ax/z^2 \exp S$	$= -2axy/z^3 \exp S$

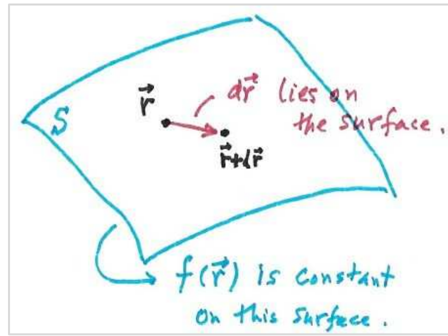
Problem 4.18

Properties of the gradient ...

Use equation (4.35) : $df = \nabla f \cdot d\mathbf{r}$

(a) Theorem: ∇f is perpendicular to any surface on which $f(\mathbf{r})$ is constant.

Proof:



Let $d\mathbf{r}$ be a small displacement ***on the surface***.

$df = f(\mathbf{r}+d\mathbf{r}) - f(\mathbf{r}) = 0$ because f is constant on S.

$\nabla f \cdot d\mathbf{r} = 0$ implies ∇f is $\perp$ to $d\mathbf{r}$.

$d\mathbf{r}$ is an arbitrary displacement on S, so ∇f is $\perp$ to S.

(b) Theorem: ∇f points in the direction in which the rate of change of $f(\mathbf{r})$ is maximum.

Proof: Consider a small displacement $\delta\mathbf{r} = \hat{\mathbf{u}}\,\delta l$;

($\hat{\mathbf{u}}$ = a unit vector and δl = an infinitesimal distance.)

Then $f(\mathbf{r}+\delta\mathbf{r}) = f(\mathbf{r}) + \nabla f \cdot \delta\mathbf{r}$ by Taylor's theorem.

$= f(\mathbf{r}) + \nabla f \cdot \hat{\mathbf{u}}\,\delta l = f(\mathbf{r}) + |\nabla f|\,\delta l\,\cos\theta$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$.

The maximum change of f occurs for $\cos\theta = 1$; that is, for $\theta = 0$.

So, the maximum of $\delta f/\delta l$ occurs if the direction vector $\hat{\mathbf{u}}$ is in the same direction as ∇f .

Problem 4.23

Is the force conservative or nonconservative ?

A conservative force has $\nabla \times \mathbf{F} = 0$, and $\mathbf{F} = -\nabla U$.

(a) $\mathbf{F} = k\{x, 2y, 3z\}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ x & 2y & 3z \end{vmatrix} = e_x . 0 + e_y . 0 + e_z . 0 = 0$$

F is conservative; $U = -\frac{1}{2} k\,(x^2 + 2y^2 + 3z^2)$.

(b) $\mathbf{F} = k\{y, x, 0\}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ y & x & 0 \end{vmatrix} = e_x . 0 + e_y . 0 + e_z . (1-1) = 0$$

F is conservative; $U = -k\,x\,y$.

(c) $\mathbf{F} = k\{-y, x, 0\}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = e_x . 0 + e_y . 0 + e_z . (1+1) = 2e_z$$

F is *not* conservative.