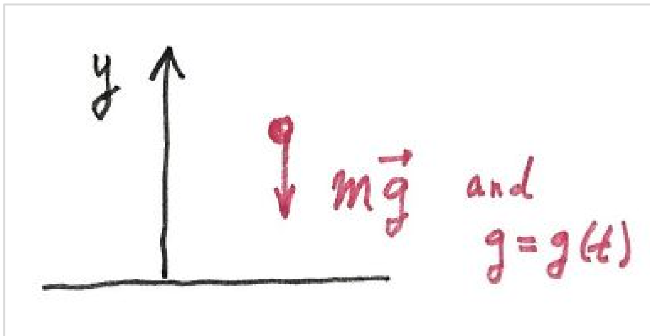
Homework Assignment #8

[37] Problem 4.26

A time-dependent gravitational field …



(a) Let U = mgy.

Then $\quad -\nabla U = -\,mg\,\mathbf{e}_y = \mathbf{F}.$

(b) Let $E = ½\,mv^2 + mgy$

Then

$$dE/dt = \underbrace{mv\,dv/dt + mg\;dy/dt}_{=\,(m\,dv/dt\,+\,mg)\,v} + m\,(dg/dt)\,y$$

= 0 by Newton's second law; $m\,dv/dt = F = -\,mg$

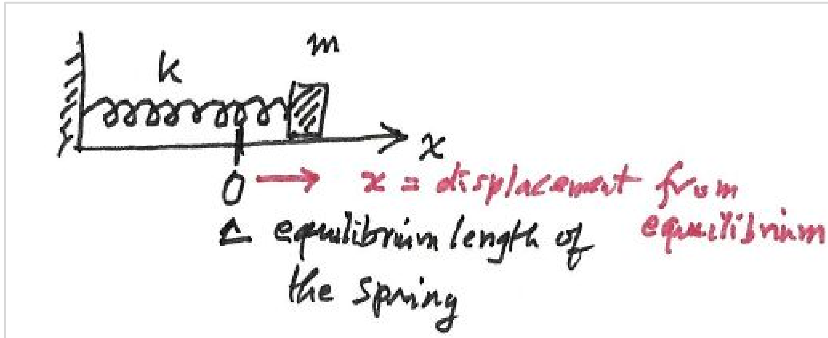
Thus $\quad dE/dt = m\,y\,(dg/dt)$ $\quad$ *which is not 0.*

[38 - part 1] Problem 4.28

A mass on a spring , using conservation of energy ...

$F = -\,k\,x$

$U = \frac{1}{2}\ k\,x^2$



Initial conditions (a sharp kick) are $x(0) = 0$ and $v(0) = v_0$;

then the mass oscillates between $x = +\,A$ and $x = -\,A$.

(a) Conservation of energy

$$E = \tfrac{1}{2}\,m\,v^2 + \tfrac{1}{2}\,kx^2 = \text{constant}$$

$$\therefore \quad v = \sqrt{2E/m \;-\; k\,x^2/m}$$

(b) The mass oscillates between $+A$ and $-A$, so $E = \frac{1}{2}\,k\,A^2$.

Thus $\quad v = \sqrt{(k/m)}\ \sqrt{(A^2 - x^2)}$

(Note : $v = 0$ at $x = \pm\,A$.)

Now,

$$dt' = \frac{dx'}{v(x')}$$

$$\int_0^t dt' = \int_0^x \frac{dx'}{v(x')}$$

$$t = \int_0^x \sqrt{\frac{m}{k}}\ \frac{dx'}{\sqrt{A^2 - x'^2}} = \sqrt{\frac{m}{k}}\ \arcsin\left(\frac{x}{A}\right)$$

(c) In the motion from $x = 0$ to $x = A$, i.e. $t = \tau/4$ (one quarter cycle)

$$t = \int_0^x \sqrt{\frac{m}{k}}\ \frac{dx'}{\sqrt{A^2 - x'^2}} = \sqrt{\frac{m}{k}}\ \arcsin\left(\frac{x}{A}\right)$$

$$t = \frac{\tau}{4} = \sqrt{\frac{m}{k}}\ \arcsin 1 = \sqrt{\frac{m}{k}}\ \frac{\pi}{2}$$

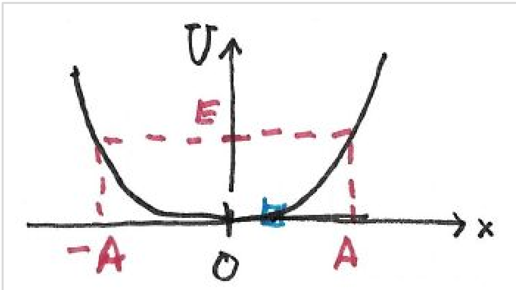
$$\therefore \text{The period is } \tau = 2\pi\sqrt{\frac{m}{k}}$$

8.3

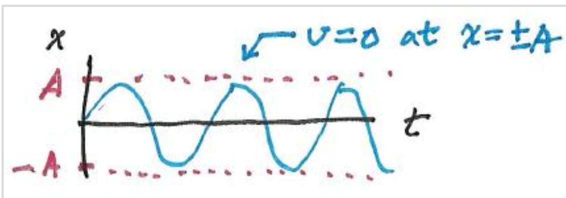
[38 - part 2] Problem 4.29

Suppose U(x) = k x^4 .

(a)



The mass oscillates between +A and −A, with energy E = k A^4 .



Conservation of energy: $kA^4 = \tfrac{1}{2} m v^2 + k x^4$

(b) The time to move from x = 0 to x = A is

$$t = \int_0^x \frac{dx'}{v(x')} = \int_0^x \frac{dx'}{\left[\frac{2k}{m}\left(A^4 - x'^4\right)\right]^{1/2}}$$

One quarter cycle:

$$\tau/4 = \sqrt{\frac{m}{2k}} \int_0^A \frac{dx'}{\sqrt{A^4 - x'^4}}$$

(c) Change the variable of integration to u = x′ /A

Note $dx' = A\,du$ and $\sqrt{A^4 - x'^4} = A^2\sqrt{1-u^4}$

$$\tau = 4\sqrt{\frac{m}{2k}}\,\frac{A}{A^2}\int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{\text{constant}}{A}$$

Thus $\tau \propto 1/A$.

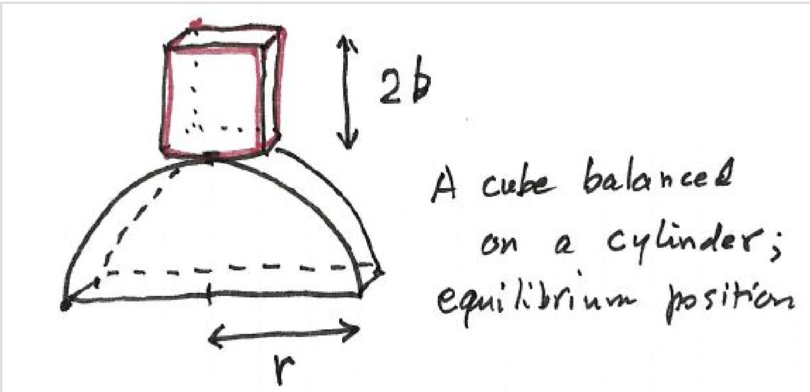
(d) Now use a computer to determine the period of oscillation, for m = k = A = 1, is

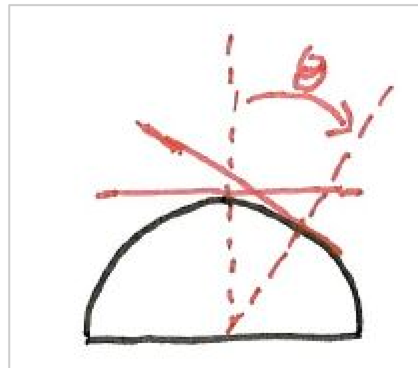
COMPUTER CALCULATION

$$\tau = \sqrt{8}\int_0^1 \frac{du}{\sqrt{1-u^4}} = 3.708$$ in time units

[39] Problem 4.33

A cube balanced on a cylinder …





(a) The potential energy is

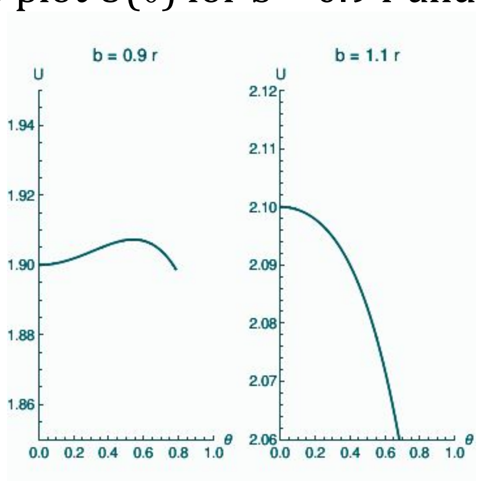
$$U(\theta) = mg\,[\,(r+b)\cos\theta + r\,\theta\,\sin\theta\,]$$

where θ = the angle of rotation of the cube

COMPUTER CALCULATION

(b) Now use a computer to plot U(θ) for b = 0.9 r and b = 1.1 r.

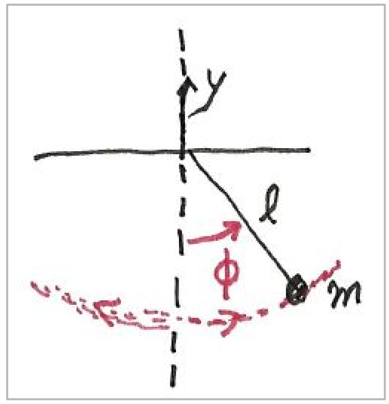
(Set r = m = g = 1.)

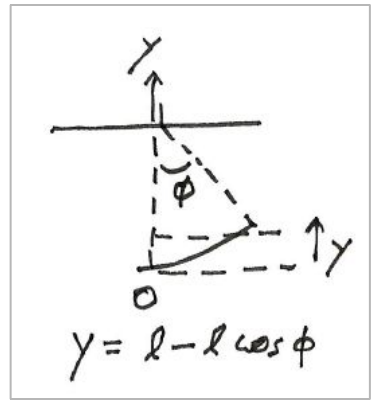


8.5

[40] Problem 4.34

The simple pendulum ...





(a) The potential energy U(φ) is

$$U(\varphi) = mg\,y = mg\,[\,\ell - \ell\cos\varphi\,] = mg\,\ell\,(1-\cos\varphi)$$

The energy is

$$E = \tfrac{1}{2} mv^2 + U = \tfrac{1}{2} m\,\ell^2 (\varphi')^2 + mg\,\ell\,(1-\cos\varphi)$$

$\boxed{\phi' \text{ means } \frac{d\phi}{dt}}$

(b) The energy is constant, so dE/dt = 0 ;

$$m\,\ell^2\,\varphi'\,\varphi'' + mg\,\ell\,\sin\varphi\;\varphi' = 0$$

Thus, $\varphi'' = -\,(g/\ell)\,\sin\varphi$

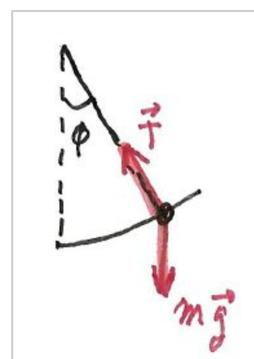
This is the same as $\boldsymbol{I}\,\alpha = \Gamma$ (torque about the suspension point)

$$I = m\ell^2 \;;\quad \alpha = \ddot{\phi}$$

$$\vec{\Gamma} = \vec{r}\times\vec{F} \;;\quad \Gamma = -\ell\, mg \sin\phi$$

$$m\ell^2\,\ddot{\phi} = -mg\ell\,\sin\phi$$

$$\ddot{\phi} = -\,g/\ell\,\sin\phi$$



(c) For small φ we can approximate sin φ ≈ φ. With that,

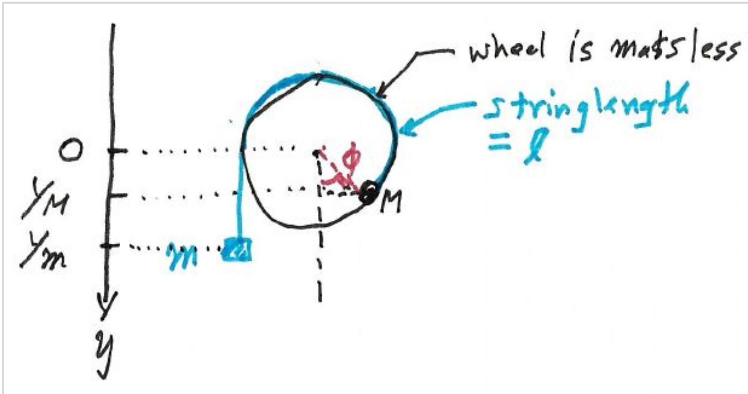
$$\varphi'' = -\,(g/\ell)\,\varphi = -\,\omega^2\varphi \quad \text{where} \quad \omega = \sqrt{g/\ell}$$

The small oscillations are harmonic, with period

$$\tau = 2\pi/\omega = 2\pi\sqrt{\ell/g}$$

[41] Problem 4.37

Wheel and two masses …



(a) The potential energy is

$U(\varphi) = -\, Mg\, y_M - mg\, y_m$

where φ = the angle of rotation of the wheel and

$y_M = R\cos\phi$; $y_m = \ell - \frac{3}{2}\pi R + R\phi$

CHECK If $\phi = \frac{\pi}{2}$ then $\pi R + y_m = \ell$;
if $\phi = 0$ then $\frac{3}{2}\pi R + y_m = \ell$;
$y_m + \pi R + (\pi/2 - \phi) R = \ell$

Thus $U = -\, Mg R \cos\varphi \; - mg R\varphi \;\; + \;\;$ *constant*

We'll choose the constant such that $U(\varphi=0) = 0$;

then, $\quad U = +\, Mg R\, (\, 1 - \cos\varphi\,) - mg R\, \varphi$

(b) The equilibrium condition is $dU/d\varphi = 0$;

$M g R \sin\varphi - m g R = 0$ implies $\sin\varphi_0 = m/M$ (*equilibrium*)

If m > M then the equilibrium is $c_0 = 3\pi/2$.

If m < M then there are two equilibrium positions:

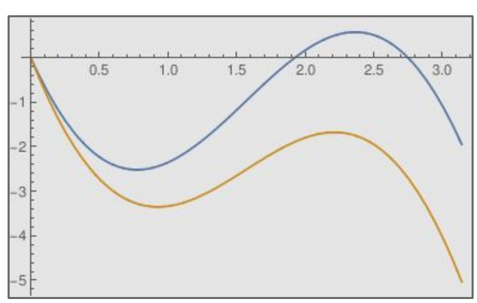
$\varphi_0 = \pi/2 \pm \psi$ where $\psi = \arccos(m/M)$

COMPUTER CALCULATION

(c) Uses a computer calculation.

Blue m = 0.7 M: φ=0 is stable;

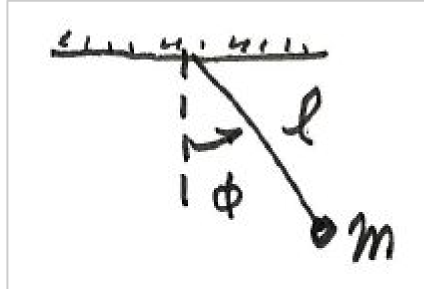
orange m = 0.8 M: φ=0 is unstable.



(d) From the computer calculation, the critical value of m/M is 0.725.

[42] Problem 4.38

The simple pendulum again, not limited to small angles …



The equation of motion:
$$\ddot{\phi} = -\,g/\ell\ \sin\phi$$

(a) Let Φ be the amplitude of oscillation; calculate τ = the period of oscillation.

- The potential energy is $U(\varphi) = mg\,\ell\,(1 - \cos\varphi)$

  and the energy is

  $E = \frac{1}{2} m\,\ell^2\,\varphi'^2 + mg\,\ell\,(1-\cos\varphi) = mg\,\ell\,(1-\cos\Phi)$

  Thus, $\varphi'^2 = (2g/\ell)\ (\cos\varphi - \cos\Phi)$

- Separation of variables: $d\varphi/dt = \varphi'$ so $dt = d\varphi/\varphi'$

$$\int_0^t dt = \int_0^\phi \frac{d\phi'}{\dot\phi} = \int_0^\phi \frac{d\phi'}{\sqrt{\frac{2g}{\ell}(\cos\phi' - \cos\Phi)}}$$

$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$

$$t = \sqrt{\frac{\ell}{2g}}\int_0^\phi \frac{d\phi'}{\sqrt{2(\sin^2\Phi/2 - \sin^2\phi'/2)}}$$

One quarter cycle:

$t = \tau/4$ and $\phi' : 0 \to \Phi$;

thus $$\frac{\tau}{4} = \sqrt{\frac{\ell}{4g}}\int_0^\Phi \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}}$$

$\frac{1}{2}\frac{\tau_0}{2\pi} = \frac{\tau_0}{4\pi}$

For small oscillations, $\tau_0 = 2\pi\sqrt{\ell/g}$

- Now change the variable of integration.

Let $u = \dfrac{\sin \phi/2}{\sin \Phi/2}$: $0 \to 1$

$u = \frac{1}{A} \sin \phi/2$ where $A = \sin \Phi/2$

$du = \frac{1}{A} \frac{1}{2} \cos \frac{\phi}{2}\, d\phi = \frac{1}{2A} \sqrt{1 - A^2 u^2}\, d\phi$

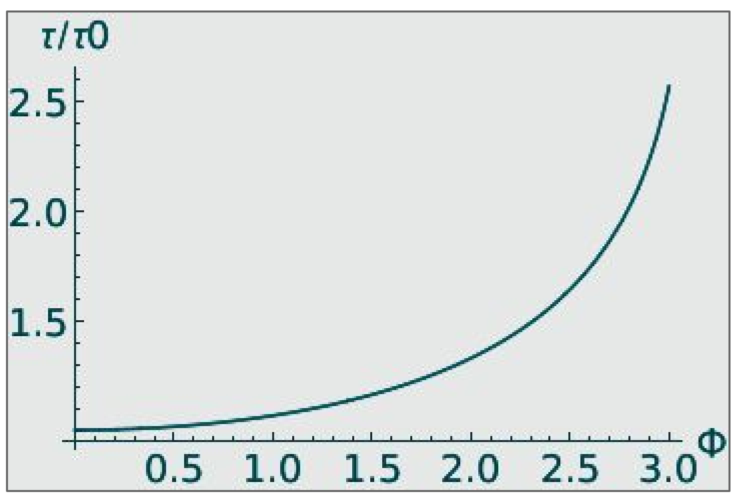
and $\sqrt{\sin^2 \Phi/2 - \sin^2 \phi/2} = \sqrt{A^2 - A^2 u^2} = A\sqrt{1-u^2}$

Thus $\tau = \dfrac{\tau_0}{\pi} \displaystyle\int_0^1 \frac{2A\, du}{\sqrt{1 - A^2 u^2}} \frac{1}{A\sqrt{1-u^2}}$

$$\tau = \frac{2\tau_0}{\pi} \underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}\,\sqrt{1 - A^2 u^3}}}_{\text{This is } K(A^3).}$$

COMPUTER CALCULATION

(c) Use Mathematica to get the values of $K(A^2)$ where $A = \sin(\Phi/2)$, and plot a graph of τ/τ_0 versus Φ.



- The limit of τ/τ_0 as Φ approaches π is ∞, because $\varphi = \pi$ is an equilibrium point of the pendulum (the inverted pendulum).