[41] Problem 4.41 (Virial theorem) $\longrightarrow T = \frac{n}{2} U$.

$$U = kr^n \Rightarrow F_r = -\frac{dU}{dr} = -knr^{n-1}$$

For circular motion, $a_r = \frac{-v^2}{r}$;

therefore
$$-\frac{v^2}{r} = \frac{F_r}{m} = -\frac{k}{m} n r^{n-1}$$

$$v^2 = \frac{k}{m} n r^n$$

The kinetic energy $T = \frac{1}{2} m v^2 = \frac{1}{2} k n r^n = \frac{n}{2} U$

Problem 4.43

(a) Given $\vec{F}(\vec{r}) = f(r)\hat{e}_r = f(r)\frac{\vec{r}}{r} = \frac{f(r)}{r}\left[x\hat{e}_x + y\hat{e}_y + z\hat{e}_z\right]$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{xf}{r} & \frac{yf}{r} & \frac{zf}{r} \end{vmatrix} = \hat{e}_x\left[\frac{\partial}{\partial y}\left(\frac{zf}{r}\right) - \frac{\partial}{\partial z}\left(\frac{yf}{r}\right)\right] + \hat{e}_y \text{ sim.} + \hat{e}_z \text{ sim.}$$

$$= \hat{e}_x\left[z\frac{d}{dr}\left(\frac{f}{r}\right)\underbrace{\frac{dr}{\partial y}}_{\frac{y}{r}} - y\frac{d}{dr}\left(\frac{f}{r}\right)\underbrace{\frac{dr}{\partial z}}_{\frac{z}{r}}\right] + \text{other}$$

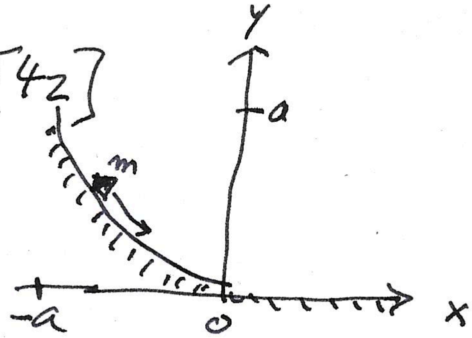
$$= \hat{e}_x (zy - yz)\frac{1}{r}\frac{d}{dr}\left(\frac{f}{r}\right) + \text{other}$$

$$= 0.$$

So $\vec{F}$ is conservative.

(b) In polar coordinates, $\nabla \times \vec{F} = 0$.

[42]



Track is $y = x^2/a$ for $x<0$.

Initial point is $(x_0, y_0) = (-a, a)$.

(A) Energy is conserved $\Rightarrow$

$$\frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy = mga$$

Here $x = \sqrt{ay}$ so $\dot{x} = \sqrt{a}\,\frac{1}{2} y^{-\frac{1}{2}} \dot{y} = \frac{1}{2}\sqrt{\frac{a}{y}}\,\dot{y}$

Thus $\frac{1}{2} m \left(\frac{1}{4}\frac{a}{y} + 1 \right) \dot{y}^2 + mgy = mga$

$$\dot{y}^2 = \frac{mg(a-y)}{\frac{m}{2}\left(\frac{a}{4y}+1\right)} = 2ga\,\frac{y(1-y/a)}{y + a/4}$$

$$\dot{y} = f(y) = -\sqrt{2ga}\left[\frac{y(1-y/a)}{y+a/4}\right]^{1/2}$$

(B) $dt = \frac{dy}{\dot{y}} \Rightarrow \int dt = \int_a^0 \frac{dy_i}{f(y_i)} = \frac{1}{\sqrt{2ga}} \int_0^a \frac{\sqrt{y + a/4}}{\sqrt{y(1-y/a)}}\,dy$

let $y = a\xi$

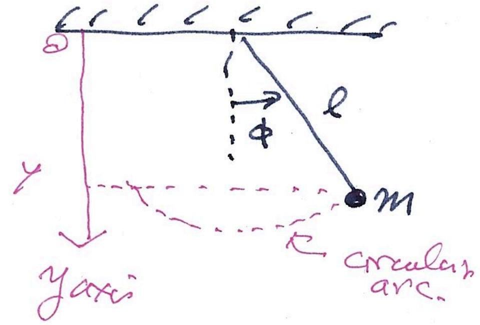
then $dy = a\,d\xi$

$$\int dt = \text{time} = \frac{1}{\sqrt{2ga}} \int_0^1 \frac{\sqrt{a}\sqrt{\xi + 1/4}\;a\,d\xi}{\sqrt{a\xi(1-\xi)}}$$

$$\text{time} = \sqrt{\frac{a}{2g}}\; \underbrace{\int_0^1 \frac{\sqrt{\xi + 1/4}\;d\xi}{\sqrt{\xi(1-\xi)}}} = 1.874\sqrt{\frac{a}{g}}$$

Use Mathematica;
integral = 2.635

[43] Problem 5.3



$$U = -mgy$$

$$y = l\cos\phi$$

$$U(\phi) = -mgl\cos\phi$$

For small ϕ, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$

$$U \approx -mgl + \frac{1}{2}mgl\,\phi^2$$

$$= U_0 + \frac{1}{2}K\phi^2 \quad \text{where} \quad K = mgl.$$

[44] Problem 5.5

Given I : $x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

$$= (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$$

$$= B_1\cos\omega t + B_2\sin\omega t \; ; \qquad \begin{aligned} B_1 &= c_1 + c_2 \\ B_2 &= i(c_1 - c_2) \end{aligned}$$

Given II : $x = B_1\cos\omega t + B_2\sin\omega t$

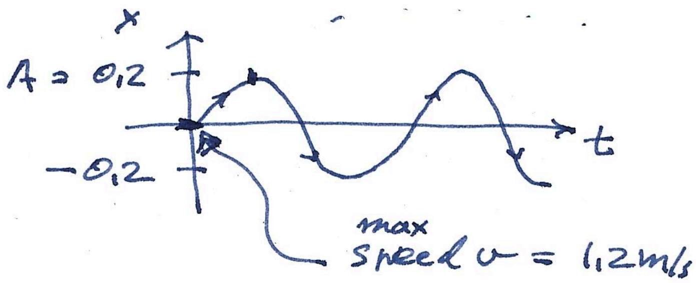
$$= A\cos\phi\cos\omega t + A\sin\phi\sin\omega t$$

$$= A\cos(\omega t - \phi) \; ; \qquad \begin{aligned} A\cos\phi &= B_1 \\ A\sin\phi &= B_2 \end{aligned}$$

Given III : $x = A\cos(\omega t - \phi)$

$$= A\,\mathrm{Re}\, e^{i\omega t}e^{-i\phi} = \mathrm{Re}(ce^{i\omega t}) \; ; \qquad c = Ae^{-i\phi}$$

[45] Problem 5.9



Energy is conserved, so $E = \frac{1}{2}mv^2 = \frac{1}{2}kA^2$.

The period is

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{A^2}{v^2}} = 1.047 \text{ sec.}$$

[46] Problem 5.12 Define $\langle f \rangle = \frac{1}{\tau}\int_0^\tau f(t)\,dt$

Say $x = A\cos\omega t$ and $v = \dot{x} = -A\omega\sin\omega t$

Then

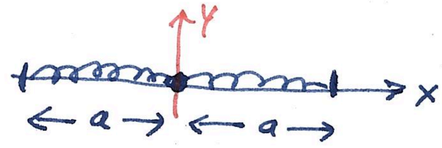
$$\langle T \rangle = \frac{1}{\tau}\int_0^\tau \frac{1}{2}mv^2\,dt = \frac{1}{\tau}\frac{m}{2}A^2\omega^2 \underbrace{\int_0^\tau \sin^2\omega t\,dt}_{=\frac{1}{2}\tau\,;}$$

$$\langle T \rangle = \frac{1}{\tau}\frac{m}{2}A^2\frac{k}{m}\frac{\tau}{2} = \frac{1}{4}kA^2 = E/2 \checkmark$$

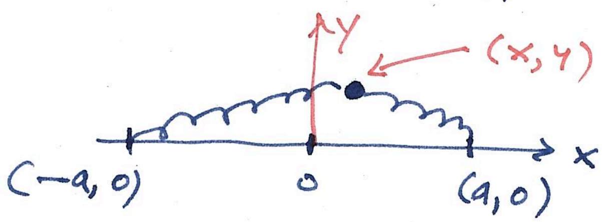
$$\langle U \rangle = \frac{1}{\tau}\int_0^\tau \frac{1}{2}kx^2\,dt = \frac{1}{\tau}\frac{k}{2}A^2 \underbrace{\int_0^\tau \cos^2\omega t\,dt}_{=\frac{1}{2}\tau}$$

$$\langle U \rangle = \frac{1}{\tau}\frac{k}{2}A^2\frac{\tau}{2} = \frac{1}{4}kA^2 = E/2 \checkmark$$

[47] Problem 5.18 EQUILIBRIUM



DISPLACED FROM EQUILIBRIUM



$$U = \tfrac{1}{2}k\left[\sqrt{(x+a)^2+y^2} - l_0\right]^2 + \tfrac{1}{2}k\left[\sqrt{(x-a)^2+y^2} - l_0\right]^2$$

where l_0 = equilibrium length

Expand in small x and y using Taylor's theorem:

$$\sqrt{1+\epsilon} \approx 1 + \tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2$$

$$\sqrt{(x+a)^2+y^2} = \sqrt{a^2+2ax+x^2+y^2} = a\sqrt{1+\underbrace{\frac{2x}{a}+\frac{x^2+y^2}{a^2}}_{\epsilon}}$$

$$\simeq a\left\{1+\frac{1}{2}\left(\frac{2x}{a}+\frac{x^2+y^2}{a^2}\right)-\frac{1}{8}\left(\frac{2x}{a}+\frac{x^2+y^2}{a^2}\right)^2\right\}$$

$$\cong a\left\{1+\frac{x}{a}+\frac{x^2+y^2}{2a^2}-\frac{x^2}{2a^2}+O(1/a^3)\right\}$$

$$= a\left\{1+\frac{x}{a}+\frac{y^2}{2a^2}+O(1/a^3)\right\}$$

Thus

$$U = \tfrac{1}{2}k\left\{l_0-a+x+\frac{y^2}{2a}\right\}^2 + \tfrac{1}{2}k\left\{l_0-a-x+\frac{y^2}{2a}\right\}^2$$

$$= \tfrac{1}{2}k(l_0-a)^2 + k(l_0-a)x + \tfrac{1}{2}kx^2 + \tfrac{1}{2}k\,2(l_0-a)\frac{y^2}{2a} + \text{neglected}$$

$$+\tfrac{1}{2}k(l_0-a)^2 - k\cdot(l_0-a)x + \tfrac{1}{2}kx^2 + \tfrac{1}{2}k\,2(l_0-a)\frac{y^2}{2a} + \text{neglected}$$

$$= k(l_0-a)^2 + kx^2 + 2k(l_0-a)\frac{y^2}{2a}$$

← Form of 5.104; anisotropic oscillations

• If $a<l_0$, i.e. the springs are compressed for $y=0$ the