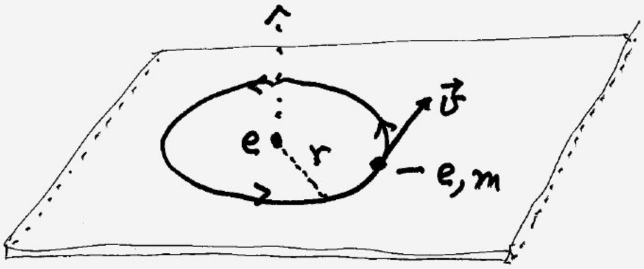
Homework Assignment #10

[47] PROBLEM 4.53 ***Scattering from a hydrogen atom …***

(a) Classical model of a hydrogen atom



Newton's second law

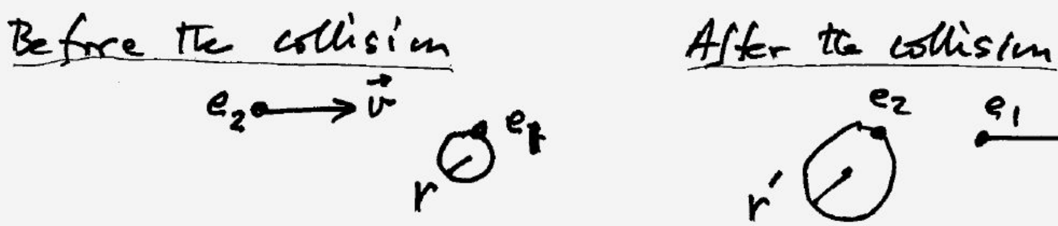
$$m a_r = F_r$$

$$-m\frac{v^2}{r} = -\frac{Ke^2}{r^2}$$

$$\frac{1}{2}mv^2 = \frac{Ke^2}{2r} \quad ; \quad U = \frac{-Ke^2}{r}$$

Thus $T = -\frac{1}{2}U$ and $E = T + U = \frac{1}{2}U$.

(b) The scattering process



In general, $E = \frac{1}{2}m v_1^2 - \frac{Ke^2}{r_1} + \frac{1}{2}m v_2^2 - \frac{Ke^2}{r^2} + \frac{e^2}{|\vec r_1 - \vec r_2|}$

(c) Conservation of energy

Energy before the collision $E = T_2 + \frac{Ke^2}{2r}$

" after the collision $E = T_1 + \frac{Ke^2}{2r'}$

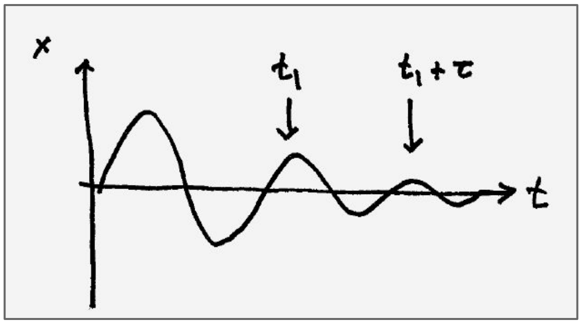
Energy is conserved, so

$$T_1 = T_2 + \frac{Ke^2}{2}\left(\frac{1}{r} - \frac{1}{r'}\right).$$

[48] PROBLEM 5.25

"Frequency" for an underdamped oscillator...

Consider a damped oscillator with $\beta < \omega_0$ (underdamped);



We can write the solution $x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$ where δ is arbitrary. W.L.O.G. set $\delta = 0$.

$x = A e^{-\beta t} \cos(\omega_1 t)$ and $v = \dot{x} = A\left[-\beta e^{-\beta t} \cos(\omega_1 t) - \omega_1 e^{-\beta t} \sin(\omega_1 t)\right]$.

(a) Let τ_1 = the time between maxima.

(a) The maxima of $x(t)$ occur when $v = 0$; that is, $\tan \omega_1 t = -\beta/\omega_1$.

$\tan(\omega_1(t_1 + \tau)) = \tan \omega_1 t \Rightarrow \omega_1 \tau = 2\pi$

Thus $\tau = 2\pi/\omega_1$.

(b) Let τ_1' = 2 x the time between zeros.

(c) Suppose $\beta = \omega_0/2$; then $\omega_1 = (\omega_0^2 - \beta^2)^{1/2} = \omega_0 \sqrt{3}/2$

(b) The zeros of $x(t)$ occur when $\cos(\omega_1 t) = 0$.

$\omega_1 t_n = \pi/2 + n\pi$

Define $\tau = t_{n+2} - t_n = \frac{1}{\omega_1}\left[\frac{\pi}{2} + (n+2)\pi - \frac{\pi}{2} - n\pi\right]$

Thus $\tau = 2\pi/\omega_1$.

(c) Suppose $\beta = \frac{\omega_0}{2}$. Then $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \frac{\sqrt{3}}{2}\omega_0$.

[49] PROBLEM 5.30 **An overdamped oscillator ...**

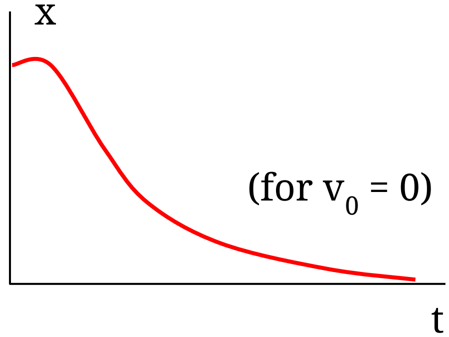
(This problem involves some computer calculations.)

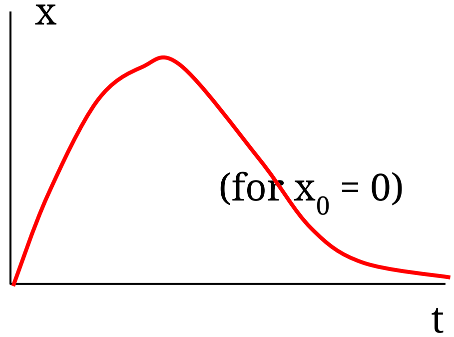
(a) $x(t) = C_1 \exp(p_1 t) + C_2 \exp(p_2 t)$ where $p_{\{1,2\}} = -\beta \pm \sqrt{(\beta^2 - \omega_0^2)}$

Solve for C_1 and C_2 from $x(0) = x_0$ and $x'(o) = v_0$

$C_1 = (p_2 x_0 + v_0) / (p_2 - p_1)$ and $C_2 = (p_1 x_0 + v_0) / (p_1 - p_2)$

(b) Sketches of the graphs of x(t) for (i) $v_0 = 0$ and for (ii) $x_{0-} = 0$:





(c)

Let $\beta \rightarrow 0$. Then $p_1 \rightarrow p_1 \rightarrow i\,\omega_0$ and $p_2 \rightarrow -i\,\omega_0$.

Then $x(t) = C1 \exp(\,i\,\omega_0\, t) + C2 \exp(-i\,\omega_0\, t\,)$

which is an undamped oscillation.

10.4

[50] PROBLEM 5.37 ***A driven underdamped oscillator …***

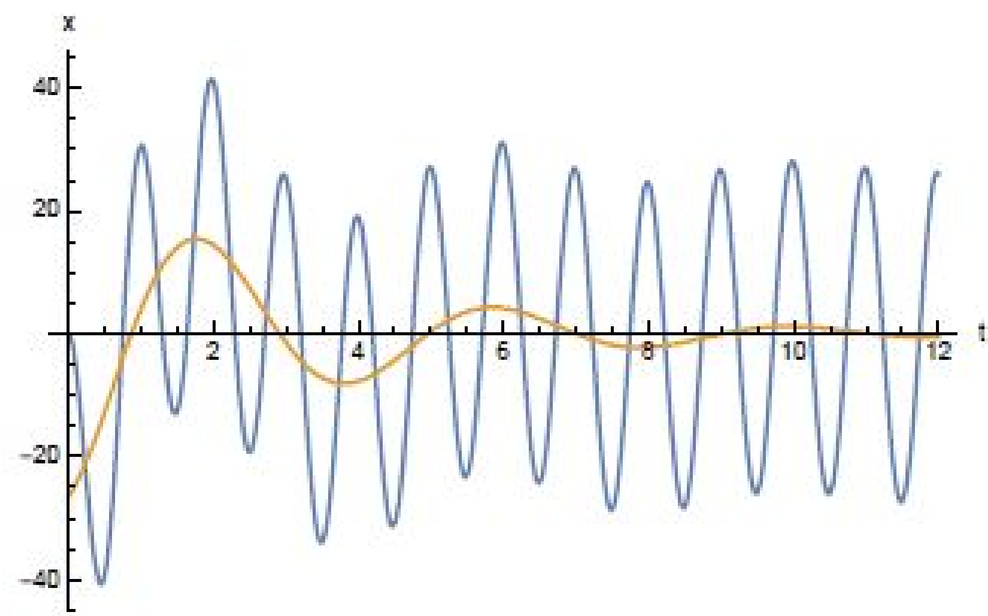
(This problem is a computer problem, based on Example 5.3.)
Consider a driven damped oscillator, with these parameter values

$\omega = 2\pi$ and $\omega_0 = \pi/2$

$\beta = 0.2\ \omega_0$ and $f_0 = 1000.$

Note that $\beta < \omega_0$, so this is an underdamped oscillator.
Initial conditions: $x_0 = 0$ and $v_0 = 0.$

Plot x(t)



Compare with Example 5.3 and explain the similarities and difference:

/1/ Problem 5.37 has transient oscillations with a longer period; $\omega_0 = \pi/2$ compared to $\omega_0 = 10\ \pi$.

/2/ Problem 5.37 has a longer decay time; $\beta = 0.1\ \pi$ compared to $\beta = 0.5\ \pi$.

/3/ Problem 5.37 and Example 5.3 have the same frequency of steady state motion; $\omega = 2\pi$ in both cases.

[51] PROBLEM 5.44

Another interpretation of the quality factor Q of a resonance …

Consider a driven oscillator with $\omega = \omega_0$. The steady state slution is

$x_p(t) = A \cos(\omega t - \delta)$ where $A = f_0/(2\beta\,\omega_0)$ and $\delta = \pi/2$.

(A) The total energy is

$E = ½\, m\, x'^2 + ½\, k\, x^2 = ½\, m\, A^2 \omega^2 \sin^2(\omega t - \delta) + ½\, m\, \omega_0^2 A^2 \cos^2(\omega t - \delta)$

$= ½\, m\, \omega^2 A^2$ (because $\omega = \omega_0$)

(B) The energy dissipated in one period is

$\Delta E = \int F_{damping}\, v\, dt = \int_0^{2\pi/\omega} m.2\beta.\, v^2\, dt = 2m\beta\, A^2\, \omega^2\, [\int \sin^2(u)\, du]\, \omega$

$= 2\,\pi\, m\, \beta\, \omega\, A^2$

(C) Now calculate $E / \Delta E$. The result is $\omega / (4\pi\beta) = Q / (2\pi)$.

Thus $Q = 2\pi\, E / \Delta E$.

[52] PROBLEM 5.52

Oscillator driven by rectangular pulses ...

(This problem is a computer problem, based on Example 5.5)

- Parameter values

  $\tau_0 = 1;\quad \omega_0 = 2\pi;\quad \beta = 0.1;$

  $\Delta\tau = 0.25;\quad f_{max} = 1;$

  $\tau = 1.0,\ 1.5,\ 2.0$ and $2.5;\quad \omega = 2\pi/\tau.$

- Fourier coefficients for $f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t)$

  $$a_0 = \frac{f_{max}\,\Delta\tau}{\tau} = \frac{0.25}{\tau}$$

  $$a_n = \frac{2 f_{max}}{\pi n} \sin\left(\frac{\pi n\,\Delta\tau}{\tau}\right) = \frac{2}{\pi n} \sin\left(\frac{\pi n}{4\tau}\right) \text{ for } n \geq 1$$

- Fourier coefficients for $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$

  $$A_n = \frac{a_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2 \omega^2}} = \frac{a_n/2\pi}{\sqrt{2\pi(1 - n^2/\tau^2)^2 + 0.04\frac{n^2}{\tau^2}}}$$

  $$\delta_n = \arctan\left[\frac{2\beta n\omega}{(2\pi)^2 - n^2\omega^2}\right]$$

- Computer program and graphs next page.

[50xx] PROBLEM 5.52

Oscillator driven by rectangular pulses ...

(This problem is a computer problem, based on Example 5.5)

Problem 5.52

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In[152]:= Remove["Global`*"]

In[173]:= Do[
  tau = 1.00000001 + 0.5 * (j - 1); omega = 2 * Pi / tau; beta = 0.1;
 a[0] = 0.25 / tau;
  Do[a[n] = 2 / (Pi * n) * Sin[Pi * n / 4 / tau], {n, 1, 10}];
  Do[A[n] = a[n] / (2 * Pi) / Sqrt[2 * Pi * (1 - n^2 / tau^2) ^2 + 0.04 * n^2 / tau^2],
   {n, 0, 11}];
  Do[delta[n] = ArcTan[(2 * beta * n * omega) / (4 * Pi^2 - n^2 * omega^2)],
   {n, 0, 11}];
  x[t_] := Sum[A[n] * Cos[n * omega * t - delta[n]], {n, 0, 10}];
  lbl = StringJoin["τ = ", ToString[tau], " τ0"];
  pl[j] = Plot[x[t], {t, 0, 7}, PlotRange → {{0, 7}, {-0.4, 0.4}},
    PlotLabel → lbl],
  {j, 1, 4}]

In[175]:= Show[GraphicsGrid[{{pl[1], pl[2]}, {pl[3], pl[4]}}], ImageSize -> Large]
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Out[175]=