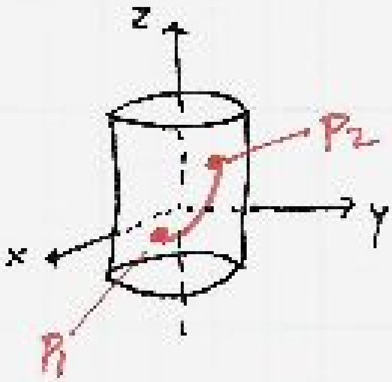
Homework Assignment #11

[51] PROBLEM 6.7 *geodesic curves on a cylinder*



On the curve, $ds = \sqrt{R^2 d\phi^2 + (dz)^2}$

$$D = \int_{P_1}^{P_2} ds = \int_{z_1}^{z_2} \sqrt{1 + R^2 \left(\frac{d\phi}{dz}\right)^2}\, dz$$

$$= \int_{z_1}^{z_2} F\, dz$$

$\delta D = 0 \Rightarrow$ the Euler Lagrange equation for $\phi(z)$

$$\frac{d}{dz}\left(\frac{\partial F}{\partial \phi'}\right) = \frac{\partial F}{\partial \phi} = 0.$$

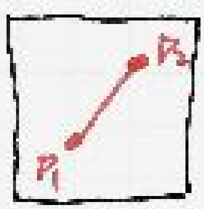
So, $\dfrac{\partial F}{\partial \phi'} = C$, a constant

$$= \left(1 + R^2 (\phi')^2\right)^{-1/2} R^2 \phi'$$

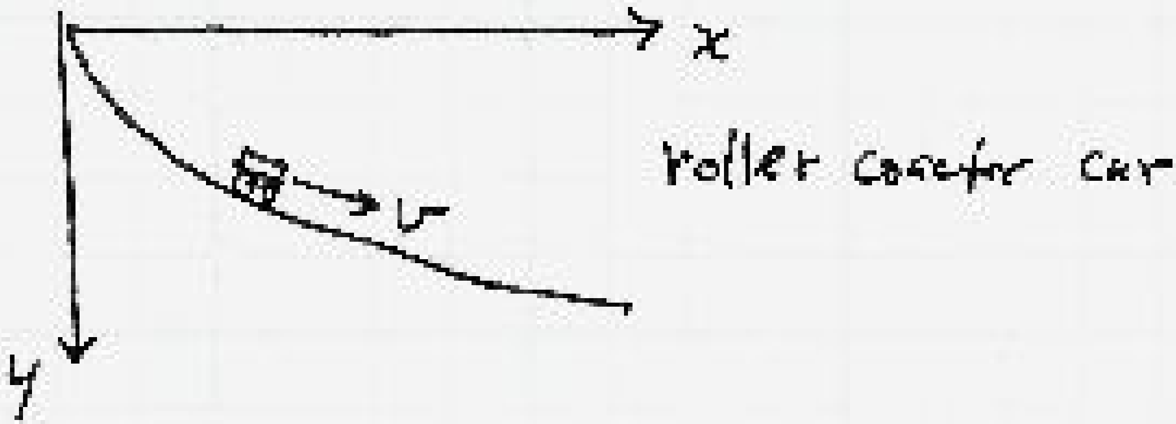
Therefore, $\phi' = \dfrac{d\phi}{dz}$ is a constant.

$$\phi(z) = \phi_1 + \frac{\phi_2 - \phi_1}{z_2 - z_1}(z - z_1)$$

If we cut the cylinder and open it out, we get



a straight line.



Energy is conserved so

$$E = \frac{1}{2} m v^2 - m g y$$

$$= E_{initial} = 0$$

Thus $\quad v = \sqrt{2 g y}$

The Euler - Lagrange equation : $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

$\boxed{6.10}$ If $\frac{\partial f}{\partial y} = 0$ then $\underbrace{\frac{\partial f}{\partial y'} = \text{constant}}_{\text{"first integral"}}$

$\boxed{6.20}$ If $\frac{\partial f}{\partial x} = 0$ then

$$\frac{df}{dx} = \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial y}y'$$

$$= \frac{\partial f}{\partial y'}y'' + \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' \quad \text{by the E.-L. equation}$$

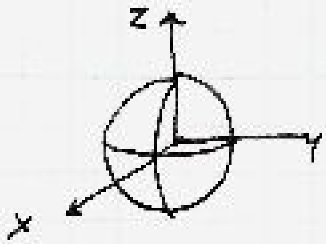
$$= \frac{d}{dx}\left[\frac{\partial f}{\partial y'} \cdot y'\right]$$

Thus $f - \frac{\partial f}{\partial y'}y' = \text{a constant}$

"first integral"

[54] PROBLEMS 6.1 AND 6.16 *geodesics on a sphere*

6.1



Using (r, θ, ϕ) the distance between 2 points is

$$ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2}$$

For 2 points on the sphere $(r = R)$

$$ds = \sqrt{R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2}$$

$$\text{Length} = \int_{\theta_1}^{\theta_2} R \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2} \, d\theta.$$

6.16 For a geodesic, $\delta L = 0$,

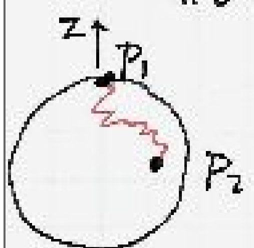
so the Euler Lagrange equation is $\dfrac{d}{d\theta}\left(\dfrac{\partial F}{\partial \phi'}\right) - \dfrac{\partial F}{\partial \phi} = 0$

for $\phi(\theta)$; $F = \sqrt{1 + \sin^2\theta \left(\frac{\partial \phi}{\partial \theta}\right)^2}$.

Since $\dfrac{\partial F}{\partial \phi} = 0$, $\dfrac{\partial F}{\partial \phi'} = $ a constant $= C$

$$\hookrightarrow \frac{\partial F}{\partial \phi'} = \left[1 + \sin^2\theta \left(\frac{\partial \phi}{\partial \theta}\right)^2\right]^{-1/2} \frac{\partial \phi}{\partial \theta} \sin^2\theta$$

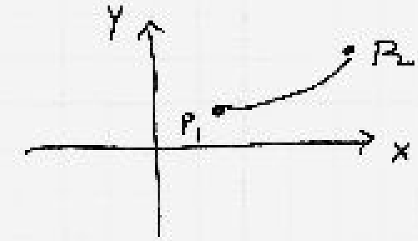
$$\sin^2\theta \frac{\partial \phi}{\partial \theta} = C \left[1 + \sin^2\theta \left(\frac{\partial \phi}{\partial \theta}\right)\right]^{1/2} \qquad (\star)$$



W.L.O.G. choose the z axis such that the point P_1 is the "NORTH POLE"; i.e. P_1 has $\theta = \theta_1 = 0$. By equation $(\star)$ $C = 0$.

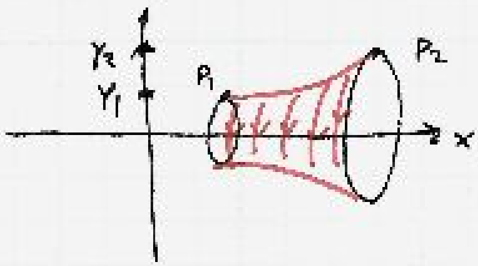
Then by $(\star)$ $\sin^2\theta \dfrac{d\phi}{d\theta} = 0$ so <u>$\phi(\theta)$ is a constant</u>.

The curve is a portion of a <u>great circle</u>; it is a "line of longitude".



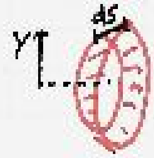
$P_1 : (x_1, y_1)$; $P_2 : (x_2, y_2)$

Curve : $x = x(y)$.



Rotate around the X axis.

To calculate the area A



$$dA = ds \cdot 2\pi y$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{(x')^2 + 1}\; dy$$

Thus

$$A = \int_{y_1}^{y_2} 2\pi y \sqrt{(x')^2 + 1}\; dy$$

Min. area $\Rightarrow \delta A = 0 \Rightarrow \dfrac{\partial f}{\partial x} = \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial x'}\right)$

(Euler Lagrange equation)

Since $\dfrac{\partial f}{\partial x} = 0$ we have

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x'}\right) = 0 \Rightarrow \frac{\partial f}{\partial x'} = \text{a constant} = m.$$

$$m = 2\pi y \frac{x'}{\sqrt{(x')^2 + 1}}$$

$\mu = m/2\pi$

$$\left(\frac{m}{2\pi}\right)^2 (x'^2 + 1) = y^2 (x')^2 \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{\mu^2}{y^2 - \mu^2}$$

Recall : $\dfrac{d}{dt} \operatorname{arccosh} t = \dfrac{1}{\sqrt{t^2 - 1}}$ $\Rightarrow$ $x - x_1 = \displaystyle\int_{y_1}^{y} \frac{\mu\, dy}{\sqrt{y^2 - \mu^2}}$ Let $y = \mu t$

$$x - x_1 = \frac{\mu^2}{\mu} \int_{y_1/\mu}^{y/\mu} \frac{dt}{\sqrt{t^2 - 1}} = \mu\left(\operatorname{arccosh} \frac{y}{\mu} - \operatorname{arccosh} \frac{y_1}{\mu}\right)$$

$x_2 - x_1 = \mu\left(\operatorname{arccosh} \dfrac{y_2}{\mu} - \operatorname{arccosh} \dfrac{y_1}{\mu}\right)$ determines μ.

$$\boxed{\therefore\; x = x_0 + y_0 \operatorname{arccosh} \frac{y}{y_0} \quad \text{where } x_0 \text{ and } y_0 \text{ are constants.}}$$

CHECK : $dx = y_0 \dfrac{1}{\sqrt{(y/y_0)^2 - 1}} \dfrac{dy}{y_0} = \dfrac{y_0\, dy}{\sqrt{y^2 - y_0^2}}$ so $y_0 = \mu$.

We have these parametric equations:

$$x = a(\theta - \sin\theta) \Rightarrow dx = a(1-\cos\theta)\,d\theta$$
$$y = a(1-\cos\theta) \Rightarrow dy = a\sin\theta\,d\theta$$

$$(dx)^2 + (dy)^2 = a^2(2-2\cos\theta)(d\theta)^2 = 2a^2(1-\cos\theta)(d\theta)^2$$

The time to move on the curve by (dx, dy) is

$$dt = \frac{ds}{v} = \frac{\sqrt{(dx)^2+(dy)^2}}{\sqrt{2g(y-y_0)}} = \frac{\sqrt{2}\,a\sqrt{1-\cos\theta}\,d\theta}{\sqrt{2ga}\sqrt{\cos\theta_0-\cos\theta}}$$

Thus

$$\text{time} = \sqrt{\frac{a}{g}}\int_{\theta_0}^{\pi}\frac{\sqrt{1-\cos\theta}\,d\theta}{\sqrt{\cos\theta_0-\cos\theta}}$$

At the bottom point, $dy = a\sin\theta\,d\theta = 0$ so $\theta=\pi$

<u>Evaluation of the integral</u>

① Let $\theta = \pi - 2\alpha$. Then $d\theta = -2\,d\alpha$. Limits: $\begin{cases}\theta : \theta_0 \to \pi \\ \alpha : \frac{1}{2}(\pi-\theta_0) \to 0\end{cases}$

$$\sqrt{1-\cos\theta} = \sqrt{1-\cos(\pi-2\alpha)} = \sqrt{1+\cos 2\alpha} = \sqrt{2\cos^2\alpha} = \sqrt{2}\cos\alpha$$
$$\sqrt{\cos\theta_0-\cos\theta} = \sqrt{\cos\theta_0 \mp 1 + 2\cos^2\alpha} = \sqrt{\cos\theta_0+1-2\sin^2\alpha}$$

② Let $u=\sin\alpha$. Then $du = \cos\alpha\,d\alpha$. Limits: $\begin{cases}\alpha : 0 \to \frac{1}{2}(\pi-\theta_0) \\ u : 0 \to \sin\left(\frac{\pi-\theta_0}{2}\right) = \cos(\theta_0/2)\end{cases}$

$$\text{time} = \sqrt{\frac{a}{g}}\int_0^{\cos(\theta_0/2)}\frac{\sqrt{2}\,2\,du}{\sqrt{\cos\theta_0+1-2u^2}}$$
$$= 2\sqrt{\frac{2a}{g}}\int_0^{\cos(\theta_0/2)}\frac{du}{\sqrt{2}\sqrt{\cos^2\frac{\theta_0}{2}-u^2}} = 2\sqrt{\frac{a}{g}}\int_0^{A}\frac{du}{\sqrt{A^2-u^2}}$$

$$\text{time} = 2\sqrt{\frac{a}{g}}\int_0^1\frac{du}{\sqrt{1-u^2}} = 2\sqrt{\frac{a}{g}}\,\frac{\pi}{2} = \pi\sqrt{a/g}$$

TAUTOCHRONE: time is independent of θ_0.