Name______grading key______________________

Homework Assignment #12 due in class Monday November 28

Staple this cover sheet in front of your solutions.

Write the requested answers on this sheet, and do the detailed solutions on your own paper.

[61] Problem 7.2 ★

Answer: Write down a *general solution* of Lagrange's equation.

$x(t) = A\cos(\omega t - \delta)$ where $\omega = \sqrt{k/m}$ 1 point

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[62] Problem 7.3 ★

Answer: Write down the solution with these initial values: $x(0) = A$, $v_x(0) = 0$ and $y(0) = 0$, $v_y(0) = B$. Prove that the trajectory is an ellipse, and sketch a graph of the trajectory.

$x = A\cos(\omega t)$ and $y = (B/\omega)\sin(\omega t)$. Note that $(x/A)^2 + (y/b)^2 = 1$ which is an ellipse.

1 point

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[63] Problems 7.8 ★★

Answer: Write general solutions for X(t) and x(t).

$X(t) = c_1 + c_2 t$ and $x(t) = A\cos(\omega t - \delta)$ where $\omega = \sqrt{2k/m}$ 2 points

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[64] Problems 7.14 ★

Answer: The so-called "crude model" does not resemble a real yo-yo at all. In a real yo-yo there are two radii − the large radius (R) of the sides and the much smaller radius (r) of the axle. Calculate the acceleration for the real yo-yo and write the result here.

$d^2x/dt^2 = 2r^2 / (R^2 + 2r^2)\, g$ 1 point

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[65] Problem 7.21 ★

Answer: If the the bead is released at time 0 with $r = R/2$ and $dr/dt = 0$, calculate the time when the bead flies off the end of the rod; R = length of the rod. Write the time here.

time = $1.317/\omega$ 1 point

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[66] Problem 7.31 ★★

Answer: Try to solve the equations with $x(t) = A\exp(i\omega t)$ and $\varphi(t) = B\exp(i\omega t)$. If possible determine ω.

The solution has $(k - m\omega^2) A = mg B$ and $M\omega^2 A = (mg - ML\omega^2) B$.

Therefore, $(k - m\omega^2) / (M\omega^2) = mg / (mg - ML\omega^2)$ 2 points

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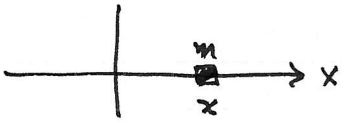
[67] Problem 7.43 ★★★ [Computer]

Answer: Hand in the computer program and the plots. 3 points

Homework Assignment #12

[61] Problem 7.2

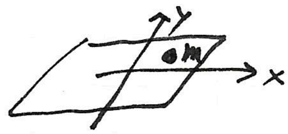


$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \Rightarrow \quad -kx = m\ddot{x}$$

Solution is $\quad x = A\cos(\omega t - \delta)$ where $\omega = \sqrt{\frac{k}{m}}$

[62] Problem 7.3



$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k(x^2 + y^2)$$

x equation is $\quad -kx = m\ddot{x} \quad \Rightarrow \quad x = A_x \cos(\omega t - \delta_x)$

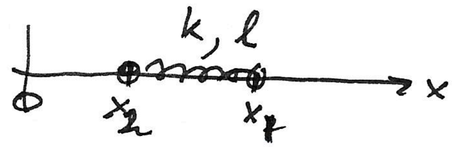
y equation is $\quad -ky = m\ddot{y} \quad \Rightarrow \quad y = A_y \cos(\omega t - \delta_y)$

Suppose $x(0) = A$, $v_x(0) = 0$; then $x = A\cos\omega t$;

and $y(0) = 0$, $v_y(0) = B$; then $y = \frac{B}{\omega}\sin\omega t$

Note $\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B/\omega}\right)^2 = 1$ so the trajectory is an ellipse.

[63] Problem 7.8



(a) $\mathcal{L} = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1 - x_2 - \ell)^2$

(b) Let $X = \frac{1}{2}(x_1 + x_2)$ (= the CM position)

and $x = x_1 - x_2 - \ell$ (= the extension)

Note $X + \frac{x}{2} = x_1 - \frac{\ell}{2}$ so $\dot{x}_1 = \dot{X} + \frac{1}{2}\dot{x}$

and $X - \frac{x}{2} = x_2 + \ell/2$ so $\dot{x}_2 = \dot{X} - \frac{1}{2}\dot{x}$

$\dot{x}_1^2 + \dot{x}_2^2 = 2\dot{X}^2 + \frac{1}{2}\dot{x}^2$

The Lagrangian is $\mathcal{L} = m\dot{X}^2 + \frac{1}{4}m\dot{x}^2 - \frac{1}{2}kx^2$

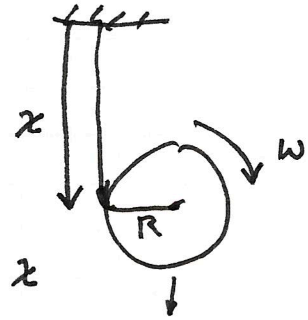
$\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\right) = 0$ implies $\ddot{X} = 0$;

$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$ implies $-kx - \frac{m}{2}\ddot{x} = 0$.

(c) § $X(t) = C_1 + C_2 t$; the CM moves with constant velocity

§ $x(t) = A\cos(\omega t - \delta)$ where $\omega = \sqrt{\frac{2k}{m}}$;
the extension undergoes simple harmonic motion.

(64) Problem 7.14



Generalized coordinate $= x$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \dot{\phi}^2 \quad \text{where} \quad I = \text{moment of inertia} = \frac{1}{2} m R^2$$

Also, $x - x_0 = R\phi$ so $\dot{\phi} = \dot{x}/R$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \cdot \frac{1}{2} m R^2 \cdot \frac{\dot{x}^2}{R^2} = \frac{3}{4} m \dot{x}^2$$

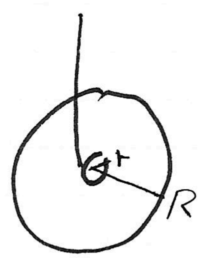
$$U = -mg\,(x - x_0)$$

$$\mathcal{L} = \frac{3}{4} m \dot{x}^2 + mgx + \text{constant}$$

Lagrange equation is $\dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}}\right)$

$$mg = \frac{3}{2} m \ddot{x} \Rightarrow \boxed{\ddot{x} = \frac{2}{3} g}$$

For a "real" yo-yo



neglect m of axle

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\phi}^2 \quad \text{where} \quad I \approx \frac{1}{2} M R^2 ;$$

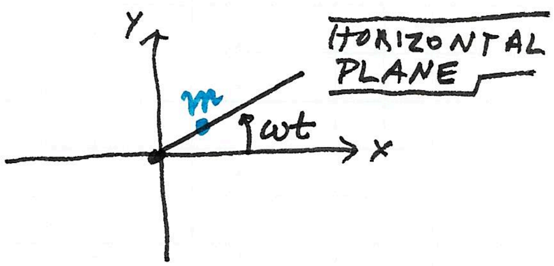
and $x - x_0 = r\phi$ so $\dot{x} = r\dot{\phi}$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} \frac{MR^2}{2} \frac{\dot{x}^2}{r^2} = \frac{1}{2}\left(M + \frac{M}{2}\frac{R^2}{r^2}\right) \dot{x}^2$$

$$U = -Mg\,(x - x_0)$$

Thus $Mg = M\left(1 + \dfrac{R^2}{2r^2}\right) \ddot{x} \Rightarrow \boxed{\ddot{x} = \dfrac{2r^2}{R^2 + 2r^2}\, g}$

[65]
Problem 7.21



$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$ where $x = r\cos\omega t$ and $y = r\sin\omega t$.

$$\left.\begin{aligned} \dot{x} &= \dot{r}\cos\omega t - r\omega\sin\omega t \\ \dot{y} &= \dot{r}\sin\omega t + r\omega\cos\omega t \end{aligned}\right\} \quad \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\omega^2$$

$U = 0$

$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2\omega^2)$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 = m\omega^2 r - m\ddot{r}$$

The equation of motion is $\ddot{r} = \omega^2 r$.

The solution is $r(t) = A e^{\omega t} + B e^{-\omega t}$

where $r(0) = A + B$ and $\dot{r}(0) = \omega(A - B)$.

Suppose $r(0) = R/2$ and $\dot{r}(0) = 0$

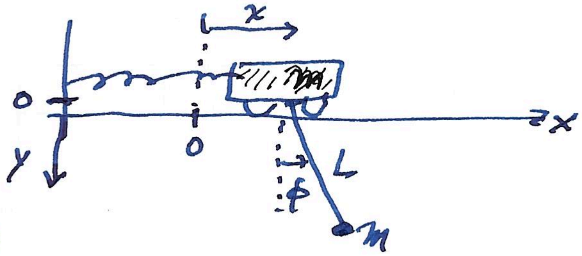
Then $r(t) = \frac{R}{4}\left(e^{\omega t} + e^{-\omega t}\right) = \frac{R}{2}\cosh(\omega t)$

The bead flies off the end of the ~~rod~~ when $r(t) = R$

The time is $\omega t = \text{arccosh}(2) = 1.317$

$$t = \frac{1.317}{\omega}$$

[66] Problem 7.31



(a) The Lagrangian (x, ϕ)

$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} M (\dot{x}_b^2 + \dot{y}_b^2) \text{ where } \begin{array}{l} x_b = x + L \sin\phi \\ y_b = L\cos\phi \end{array}$$

$$T = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} M \left[(\dot{x} + L\dot{\phi}\cos\phi)^2 + (-L\dot{\phi}\sin\phi)^2 \right]$$

$$= \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} M \left[\dot{x}^2 + L^2\dot{\phi}^2 + 2L\dot{x}\dot{\phi}\cos\phi \right]$$

And $U = -mgy + \tfrac{1}{2} k x^2 = -mgL\cos\phi + \tfrac{1}{2} k x^2$

$$\mathcal{L} = \tfrac{1}{2}(m+M)\dot{x}^2 + \tfrac{1}{2} M L^2 \dot{\phi}^2 + ML\dot{x}\dot{\phi}\cos\phi + mgL\cos\phi \mp \tfrac{1}{2} k x^2$$

(b) Assume x and ϕ are small. Then approximate

$$\mathcal{L} \approx \tfrac{1}{2}(m+M)\dot{x}^2 + \tfrac{1}{2} M L^2 \dot{\phi}^2 + ML\dot{x}\dot{\phi} + mgL\left[1 - \tfrac{1}{2}\phi^2\right] \mp \tfrac{1}{2} k x^2$$

Lagrange's equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \Rightarrow \quad -kx = (m+M)\ddot{x} + ML\ddot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \Rightarrow \quad -mgL\phi = ML^2\ddot{\phi} + ML\ddot{x}$$

Now try $x(t) = Ae^{i\omega t}$ and $\phi(t) = Be^{i\omega t}$

$$-kA = -(m+M)\omega^2 A - ML\omega^2 B = -m\omega^2 A - M\omega^2 A - ML\omega^2 B$$

$$-mgLB = -ML^2\omega^2 B - ML\omega^2 A \quad \text{or} \quad -mgB = -ML\omega^2 B - M\omega^2 A$$

$$\begin{cases} -kA = -m\omega^2 A - mg\,B \\ M\omega^2 A = mgB - ML\omega^2 B \end{cases}$$

MATRIX FORM

$$\begin{bmatrix} -k + m\omega^2 & mg \\ M\omega^2 & -mg + ML\omega^2 \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

In order to have a nontrivial solution, the determinant must equal 0

$$(m\omega^2 - k)(ML\omega^2 - mg) - mgM\omega^2 = 0$$

$$mML\,\omega^4 + \left(-kML - m^2 g - mgM\right)\omega^2 + kmg = 0$$

$$\omega^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-B}{2A} \pm \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}}$$

$$\omega^2 = \frac{kML + mg(M+m)}{2\,mML} \pm \sqrt{(\ldots)^2 - \frac{kmg}{mML}}$$

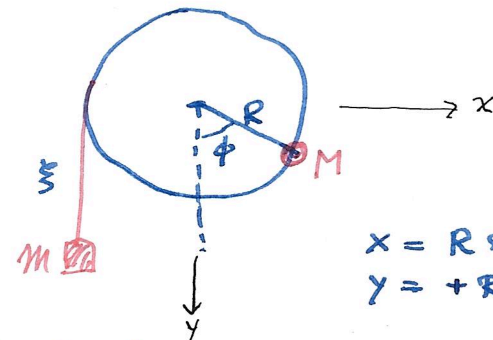
$$\omega^2 = \frac{k}{2m} + \frac{g}{2L}\left(\frac{M+m}{M}\right) \pm \sqrt{\left(\frac{k}{2m} + \frac{g}{2L}\left(\frac{M+m}{M}\right)\right)^2 - \frac{k\,g}{M\,L}}$$

For example, consider these numerical values (in appropriate units) from Problem 11.14

$$m = M = L = g = 1 \quad \text{and} \quad k = 2$$

$$\omega^2 = 1 + 1 \pm \sqrt{(1+1)^2 - 2} = 2 \pm \sqrt{2}$$

[67] Problem 7.43



Initial conditions : $\phi = 0$
and $\dot{\phi} = 0$.

$x = R \sin\phi$
$y = +R\cos\phi$

(a) $\mathcal{L} = T - U$ where $T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m\dot{\xi}^2$

$\dot{\xi} = R\dot{\phi}$
$\xi = \xi_0 + R\phi$

$T = \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}mR^2\dot{\phi}^2 = \frac{1}{2}(M+m)R^2\dot{\phi}^2$

$U = -Mgy - mg\xi = -MgR\cos\phi - mgR\phi + \text{const.}$

$\mathcal{L} = \frac{1}{2}(M+m)R^2\dot{\phi}^2 + MgR\cos\phi + mgR\phi$

Equation of motion : $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$

$\Rightarrow (M+m)R\ddot{\phi} = -Mg\sin\phi + mg$

Equilibrium points : $\ddot{\phi} = 0$ implies $\sin\phi = \frac{m}{M}$
If $m < M$ then there is an equilibrium at $\arcsin(m/M)$.

(b) Plot $U(\phi)$ for $m < M$. ← Computer Plot #1

(c) Pick these parameter values : $M = g = R = 1$, $m = 0.7$.
Solve the equation of motion of $0 \le t \le 20$. ← COMPUTER PLOT #2

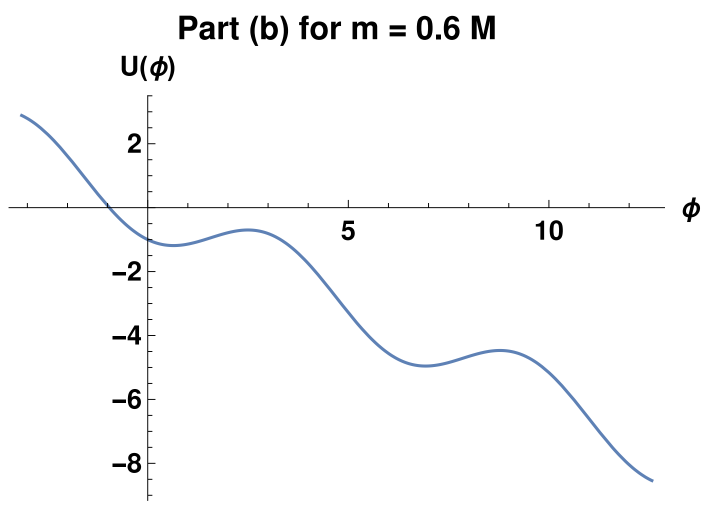
(d) Same for $m = 0.8$ ← COMPUTER PLOT #3

Problem 7.43

In[14]:=
```
bs = {FontFamily → "Helvetica", FontSize → 14, FontWeight → "Bold"}
```
Out[14]=
```
{FontFamily → Helvetica, FontSize → 14, FontWeight → Bold}
```

In[95]:=
```
(* b *)
(* Plot U(ϕ) for m < M *)
{M, g, R, m} = {1, 1, 1, 0.6};
U[ϕ_] := -M * g * R * Cos[ϕ] - m * g * R * ϕ
Plot[U[ϕ], {ϕ, -Pi, 4 Pi}, AxesLabel → {"ϕ", "U(ϕ)"},
 PlotLabel → "Part (b) for m = 0.6 M",
 BaseStyle → bs]
```

Out[97]=

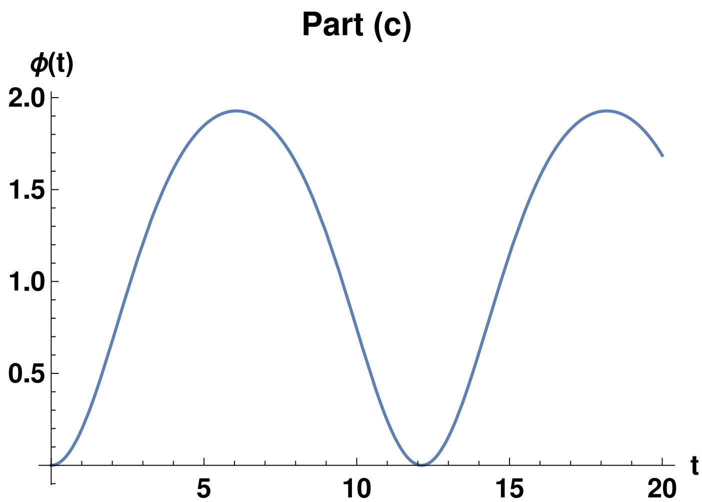


In[74]:=
```
(* c *)
? NDSolve
```

NDSolve[*eqns*, *u*, {*x*, x_{min}, x_{max}}] finds a numerical solution to the ordinary differential equations *eqns* for the function *u* with the independent variable *x* in the range x_{min} to x_{max}.
NDSolve[*eqns*, *u*, {*x*, x_{min}, x_{max}}, {*y*, y_{min}, y_{max}}] solves the partial differential equations *eqns* over a rectangular region.
NDSolve[*eqns*, *u*, {*x*, *y*} ∈ Ω] solves the partial differential equations *eqns* over the region Ω.
NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}, {*x*, *y*} ∈ Ω] solves the time-dependent partial differential equations *eqns* over the region Ω.
NDSolve[*eqns*, {u_1, u_2, …}, …] solves for the functions u_i. ≫

In[70]:=
```
{M, g, R, m} = {1, 1, 1, 0.7};
eqs = {(M + m) * R * phi''[t] == -M * g * Sin[phi[t]] + m * g,
   phi[0] == 0, phi'[0] == 0};
RR = NDSolve[eqs, phi, {t, 0, 20}];
angle = phi /. RR[[1]];
Plot[angle[t], {t, 0, 20}, AxesLabel → {"t", "ϕ(t)"},
 PlotLabel → "Part (c)",
 BaseStyle → bs]
```

Out[73]=



In[90]:=
```
(* d *)
{M, g, R, m} = {1, 1, 1, 0.8};
eqs = {(M + m) * R * phi''[t] == -M * g * Sin[phi[t]] + m * g,
   phi[0] == 0, phi'[0] == 0};
RR = NDSolve[eqs, phi, {t, 0, 20}];
angle = phi /. RR[[1]];
Plot[angle[t], {t, 0, 20}, AxesLabel → {"t", "ϕ(t)"},
 PlotLabel → "Part (d)",
 BaseStyle → bs]
```

Out[94]=