Homework Assignment #13

[71] Problem 8.4

The Lagrangian is $\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$

and Lagrange's equation is $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0.$

For $q = x$, $\frac{d}{dt}(\mu \dot{x}) + \frac{\partial U}{\partial x} = \mu \ddot{x} + \frac{\partial U}{\partial x} = 0$

For $q = y$, $\mu \ddot{y} + \frac{\partial U}{\partial y} = 0$

For $q = z$, $\mu \ddot{z} + \frac{\partial U}{\partial z} = 0$.

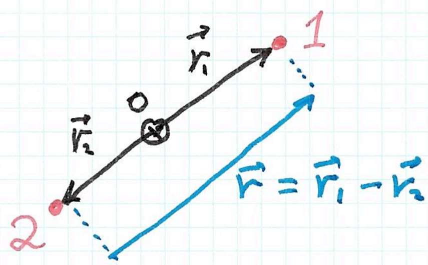
Thus

$$\mu \ddot{\vec{r}} + \nabla U = 0$$

which is the same as a particle of mass μ w/ potential energy $U(\vec{r})$.

[72] Problem 8.6

The CM frame of reference



$$\vec{r}_1 = \frac{m_2}{M}\vec{r}$$

$$\vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

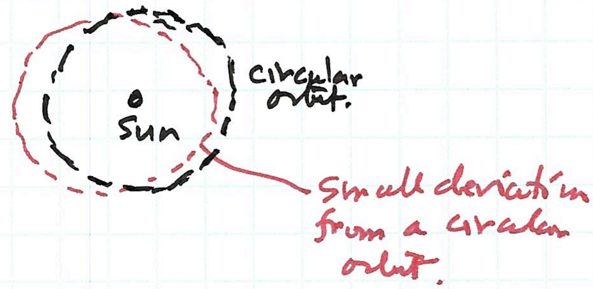
$$\vec{l}_1 = \vec{r}_1 \times m_1 \dot{\vec{r}}_1 = m_1\left(\frac{m_2}{M}\right)^2 \vec{r}\times\dot{\vec{r}} = \frac{m_2}{M}\mu\vec{r}\times\dot{\vec{r}}$$

$$\vec{l}_2 = \vec{r}_2 \times m_2 \dot{\vec{r}}_2 = m_2\left(\frac{m_1}{M}\right)^2 \vec{r}\times\dot{\vec{r}} = \frac{m_1}{M}\mu\vec{r}\times\dot{\vec{r}}$$

$$\vec{L} = \vec{l}_1 + \vec{l}_2 = \mu\vec{r}\times\dot{\vec{r}}$$

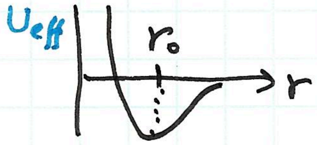
Thus $\vec{l}_1 = \frac{m_2}{M}\vec{L}$ and $\vec{l}_2 = \frac{m_1}{M}\vec{L}$, as claimed.

[73] Problem 8.12



(a) For the circular orbit

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2}$$



$$U'_{eff}(r) = \frac{Gm_1m_2}{r^2} - \frac{l^2}{\mu r^3}$$

$$U'_{eff}(r_0) = 0 \Rightarrow \boxed{r_0 = \frac{l^2}{\mu G m_1 m_2}}$$

(b) Stability of the circular orbit.

Consider $r(t) = r_0 + \epsilon(t)$ where ϵ is small.

$$U''_{eff}(r_0) = \frac{-2Gm_1m_2}{r_0^3} + \frac{3l^2}{\mu r_0^4} = \frac{1}{r_0^3}\left[-2Gm_1m_2 + \frac{3l^2}{\mu}\frac{\mu G m_1 m_2}{l^2}\right]$$

$$= \frac{Gm_1m_2}{r_0^3}$$ which is positive; so the circular orbit is stable.

Small radial oscillations.

$\mu\ddot{r} + \frac{\partial U_{eff}}{\partial r} = 0$ where $r(t) = r_0 + \underbrace{\epsilon(t)}_{\text{small}}$

$$U_{eff}(r_0+\epsilon) \approx U_{eff}(r_0) + \epsilon \underbrace{U'_{eff}(r_0)}_{=0} + \frac{1}{2}\epsilon^2 \underbrace{U''_{eff}(r_0)}_{Gm_1m_2/r_0^3}$$

So $\mu\ddot{\epsilon} + \epsilon \frac{Gm_1m_2}{r_0^3} = 0$

$\ddot{\epsilon} + \omega_R^2 \epsilon = 0 \Rightarrow$ radial oscillations have $\omega_R^2 = \frac{Gm_1m_2}{\mu r_0^3}$

The angular velocity is $\omega_\phi = \frac{d\phi}{dt} = \frac{l}{\mu r_0^2}$.

$$\frac{\omega_R}{\omega_\phi} = \sqrt{\frac{Gm_1m_2}{\mu r_0^3}}\,\frac{\mu r_0^2}{l} = \sqrt{\frac{Gm_1m_2}{\mu r_0^3}}\,\frac{\mu r_0^2}{\sqrt{r_0 \mu G m_1 m_2}} = 1$$

$\therefore$ The period of radial oscillations is equal to the period of revolutions of the planet.

[74] Problem 8.15

Kepler's third law states $\tau^2 = C a^3$ for any planet, where C is a constant. It is approximately true, and $C \approx \frac{4\pi^2}{GM_{sun}}$.

But more accurately, $C = \frac{4\pi^2}{G(M_{sun} + m)}$

where m = mass of the planet.

The planet masses range from Jupiter, $m_J = 2\times10^{27}$ kg to much smaller masses (e.g., Earth, $m_E = 6\times10^{24}$ kg).

So the variations of C are of order

$$\frac{C_E}{C_J} = \frac{M_S + m_J}{M_S + m_E} \approx 1 + \frac{m_J}{M_S} = 1 + \frac{2\times10^{27}\text{ kg}}{2\times10^{30}\text{ kg}}$$

$$= 1.001$$

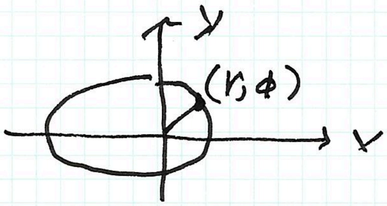
Variation $\sim 0.001 = 0.1$ percent

[75] Problem 8.16

In polar coordinates, the equation for a Keplerian orbit is $r(\phi) = \dfrac{c}{1+\varepsilon\cos\phi}$ where $c>0$ and $\varepsilon \geq 0$.

Consider a bounded orbit: $0 \leq \varepsilon < 1$.

Write the orbit equation in Cartesian coordinates (x,y).



$x = r\cos\phi$ and $y = r\sin\phi$

We have $r = \dfrac{c}{1+\varepsilon x/r}$

- $r + \varepsilon x = c$
- $r = c - \varepsilon x$
- $r^2 = x^2+y^2 = (c-\varepsilon x)^2$

  $= c^2 - 2c\varepsilon x + \varepsilon^2 x^2$
- $(1-\varepsilon^2)x^2 + 2c\varepsilon x + y^2 = c^2$
- $(1-\varepsilon^2)\left[x + \dfrac{c\varepsilon}{1-\varepsilon^2}\right]^2 + y^2$

  $= c^2 + \dfrac{c^2\varepsilon^2}{1-\varepsilon^2} = \dfrac{c^2}{1-\varepsilon^2}$
- Thus

$$\frac{\left(x+\frac{c\varepsilon}{1-\varepsilon^2}\right)^2}{c^2/(1-\varepsilon^2)^2} + \frac{y^2}{c^2/(1-\varepsilon^2)} = 1$$

Result

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$d = \dfrac{c\varepsilon}{1-\varepsilon^2}$ or $d = \varepsilon a$

$a^2 = \dfrac{c^2}{(1-\varepsilon^2)^2}$ or $a = \dfrac{c}{1-\varepsilon^2}$

$b^2 = \dfrac{c^2}{1-\varepsilon^2}$ or $b = \dfrac{c}{\sqrt{1-\varepsilon^2}}$