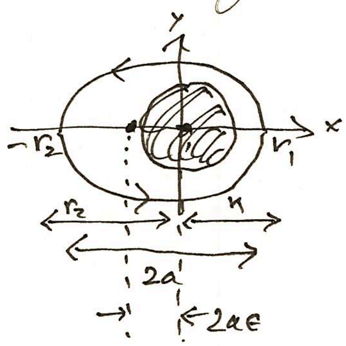
Homework Assignment 14

[MM] Problem 8.19

Radius of Earth $= R = 6400\,km$



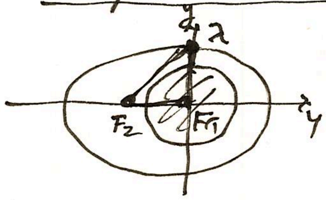
Satellite orbit : $r_1 =$ perigee $= R + 300\,km$

$r_2 =$ apogee $= R + 3000\,km$

Major axis $= 2a = r_1 + r_2 = 9700\,km$

Eccentricity $= \dfrac{2a - 2r_1}{2a} = \dfrac{r_2 - r_1}{r_1 + r_2} = 0.168$

Height at $x = 0 = \lambda$



Eq. of an ellipse : $\sqrt{y_\lambda^2} + \sqrt{(-2a\varepsilon)^2 + y_\lambda^2} = 2a$

$$\sqrt{(2a\epsilon)^2 + y_\lambda^2} = 2a - y_\lambda$$

$$(2a\epsilon)^2 + y_\lambda^2 = 4a^2 - 4a y_\lambda + y_\lambda^2$$

$$y_\lambda = \frac{4a^2(1-\varepsilon^2)}{4a} = a(1-\varepsilon^2) \quad \text{SEMI-LATUS RECTUM}$$

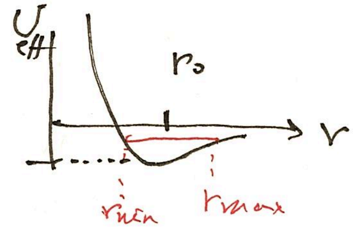
$$y_\lambda = 7824\ km$$

The height above Earth's surface $= y - R = \boxed{1424\ km}$

[M8] Problem 8.25 $\quad F = \frac{-k}{r^{5/2}}$; $\Rightarrow U = \frac{-2k}{3r^{3/2}}$

Solve the orbit equation.
Limit the parameters like this: $m=1$, $l=1$, $k=1$.

(a) $U_{eff} = \frac{l^2}{2mr^2} - \frac{2k}{3r^{3/2}}$



The minimum occurs at

$$U'_{eff}(r_0) = 0 = \frac{-2}{2r_0^3} + \frac{1}{r_0^{5/2}} = 0 \text{ ; thus } r_0 = 1.$$

(b) Consider an orbit with $E = -0.1$.

$\exists$ Turning points at r_{min} and r_{max} where

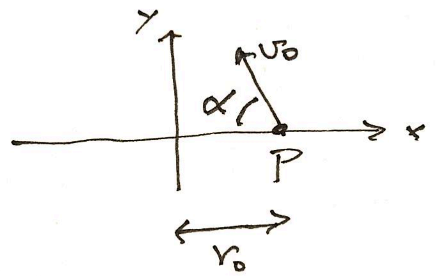
$U(r_{min}) = U(r_{max}) = E$; thus $\boxed{r_{min} = 0.6671}$

$$\boxed{\frac{1}{2r_{min}^2} - \frac{2}{3r_{min}^{3/2}} = -0.1 \Rightarrow \frac{1}{2r_{min}^2} - \frac{2}{3r_{min}^{3/2}} = -0.1}$$

(c) $u''(\phi) = -u - \frac{m}{l^2u^2}F = -u + \frac{mk}{l^2}u^{1/2}$

Solve this by Mathematica and plot the orbit.

[79] Problem 8.27 For a Keplerian orbit...



Given these initial conditions, determine the orbit parameters $\epsilon, \mathcal{E}, \delta$ where

$$r(\phi) = \frac{c}{1+\epsilon\cos(\phi-\delta)}$$

• We know $\ell^2 = \gamma\mu c = GM\mu^2 c$ is constant.

For the initial values, $\ell = \mu r_0 v_0 \sin\alpha$

Thus $GM\mu^2 c = \mu^2 r_0^2 v_0^2 \sin^2\alpha$

$$c = \frac{r_0^2 v_0^2 \sin^2\alpha}{GM}$$

$\therefore c = 8.908 \times 10^{10}$ m

for the given numerical values ↗

NUMERICAL VALUES
$r_0 = 100 \times 10^6$ km
$v_0 = 45$ km/s
$\alpha = 50$ degrees
$GM = 13.34 \times 10^{19}\ m^3 s^{-2}$

• At point P, $v_r = -v_0\cos\alpha$ and $\phi = 0$; $\ell = \mu r^2\dot\phi$ so $\dot\phi = \frac{\ell}{\mu r_0^2}$ at P

$$\therefore v_r = (\dot r)_p = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{-c\,(-\epsilon\sin(-\delta))}{[1+\epsilon\cos(0-\delta)]^2}\cdot\frac{\ell}{\mu r_0^2}$$

$$-v_0\cos\alpha = \frac{-c\,\epsilon\sin\delta}{[1+\epsilon\cos\delta]^2}\,\frac{\mu r_0 v_0\sin\alpha}{\mu r_0^2} \quad \text{and} \quad r_0 = \frac{c}{1+\epsilon\cos\delta}$$

$$v_0\cos\alpha = \frac{1}{c}\,\epsilon\sin\delta\sin\alpha\, r_0 v_0 \quad \text{and} \quad c = \frac{r_0^2 v_0^2\sin^2\alpha}{GM}$$

$$\therefore \epsilon\sin\delta = \frac{r_0 v_0^2 \sin\alpha\cos\alpha}{GM} = 0.7475$$

• At point P, $r = r_0 = \dfrac{c}{1+\epsilon\cos\delta}$

$$\therefore \epsilon\cos\delta = -1 + \frac{c}{r_0} = -1 + \frac{r_0 v_0^2\sin^2\alpha}{GM} = -0.1092$$

$\epsilon = 0.7554$ and $\delta = 1.716$ radians $= 98.3$ deg.

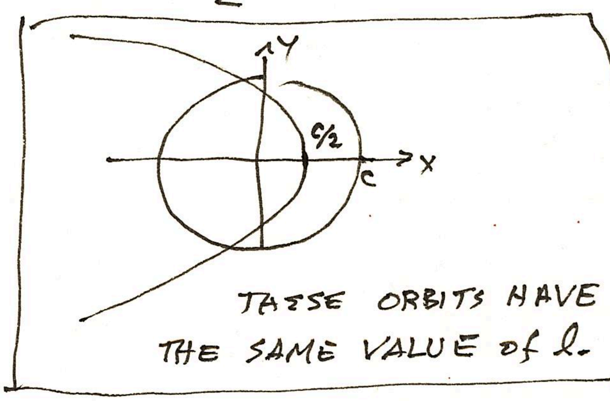
[80] Problem 8.28

The general orbit equation is $r(\phi) = \frac{c}{1+\epsilon\cos\phi}$

where $c = \gamma\mu l^2 = GM\mu^2 l^2$.

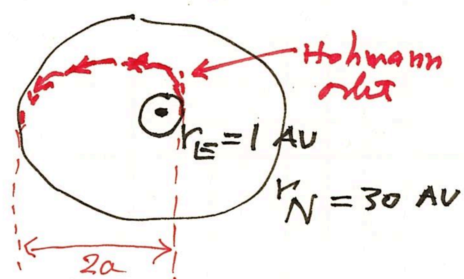
- For the circular orbit, $(\epsilon = 0)$ $\quad r_{circle} = c$
- For the parabolic orbit, $(\epsilon = 1)$ $\quad r_{min} = \frac{c}{2}$

Thus $r_{min} = \frac{1}{2} r_{circle}$



[81] = Problem 8.34

Send a spacecraft to Neptune, on a Hohmann transfer orbit



The transfer orbit is an ellipse, with $2a = r_E + r_N = 31$ AU.

By Kepler's 3rd law,

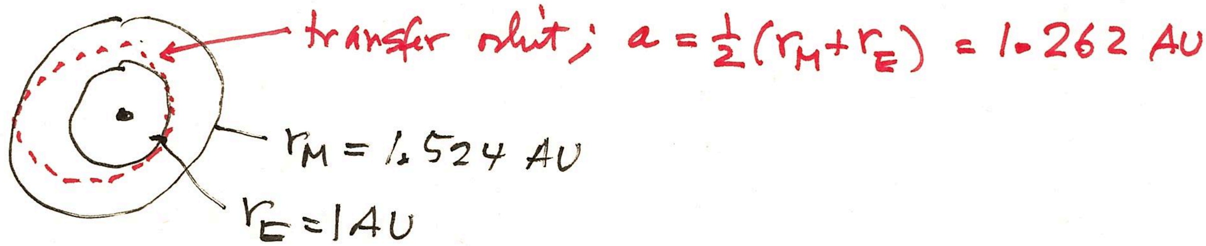
$$\tau = \sqrt{\frac{4\pi^2 a^3}{GM}} \propto a^{3/2}$$

Thus $$\frac{\tau}{\tau_{Earth}} = \left[\frac{a}{a_{Earth}}\right]^{3/2} = \frac{\tau}{1\text{ year}} = \left[\frac{15.5}{1}\right]^{3/2}$$

$$\tau = 61 \text{ years}$$

The travel time (Earth → Neptune) is $\frac{\tau}{2} = 30.5$ years.

[82] Send a satellite from Earth to Mars on a Hohmann transfer orbit.



By Kepler's 3rd law, $\frac{\tau}{\tau_{Earth}} = \left(\frac{a}{a_{Earth}}\right)^{3/2} = (1.262)^{3/2}$

$\tau = 1.418$ years

The travel time (Earth $\rightarrow$ Mars) is $\frac{1}{2}\tau$,

so travel time $= 0.709$ year $=$ 259 days