Chapter 2.
Projectiles; and Charged Particles

projectiles with air resistance

- ❑ 2.1. Air resistance
- ❑ 2.2. Linear air resistance
- ❑ 2.3. Trajectory and Range
- ❑ 2.4. Quadratic air resistance

charged particle in a magnetic field

- ❑ 2.5. Charge in B field
- ❑ 2.6. Complex Exponentials
- ❑ 2.7. Solve q in B

2.1 - Aerodynamic forces

When an object moves through air, it experiences a force.

The force is exerted by the air on the object; the reaction force is exerted by the object on the air.

The force on the object can be resolved into two components:

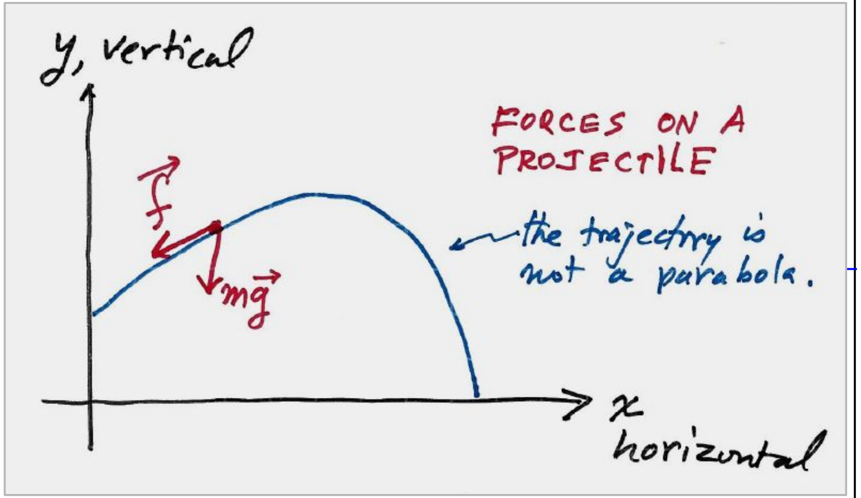
"drag" = component of force
in the direction of $-\boldsymbol{v}$

"lift" = component of force
in the direction of $-\boldsymbol{g}$

Air resistance

We will only consider the drag force; denote it by $\boldsymbol{f}$.

Figure 2.1
Motion of a projectile



The force of air resistance $\quad f(v)$

- The direction of $\boldsymbol{f}$ is parallel to $-\boldsymbol{v}$.
- The magnitude of $\boldsymbol{f}$ depends on v (speed) and on other properties of the object.
- We'll write $\boldsymbol{f} = f(v)\ (-\boldsymbol{e}_v)$;
- *and we'll write the magnitude as*

$$f(v) = \mathrm{b}\ \mathrm{v} + \mathrm{c}\ \mathrm{v}^2 = f_{\mathrm{lin}} + f_{\mathrm{quad}}$$

$$\boldsymbol{f} = f(v)\ (-\boldsymbol{e}_v)$$

$$f(v) = b\,v + c\,v^2 = f_{lin} + f_{quad}$$

- $f_{lin} = b\,v$ comes from ***viscosity;***

  for a sphere,

  $b = \beta D$ (D = diameter)

  $\beta = 3\pi\,\eta$ (η = viscosity)

- $f_{quad} = c\,v^2$ comes from ***the inertia of air;***

  for a sphere,

  $c = 0.25\,\rho\,A = \gamma\,D^2$

  $\gamma \propto \rho$ (ρ = density)

Example 2.1 BASEBALLS AND LIQUID DROPS

↳↳ comparing the relative importance of f_{quad} and f_{lin} ; consider 3 cases.

For a sphere moving through air at STP,

$f_{lin} = \beta D v$ and $\beta = 1.6 \times 10^{-4}\ Ns/m^2$

$f_{quad} = \gamma D^2 v^2$ and $\gamma = 0.25\ Ns^2/m^4$

in MKS units.

	D	v [m/s]	f_{quad} / f_{lin}	dominant
1. baseball	7 cm	5	600	cv^2
2. small raindrop	1 mm	0.6	1	comparable
3. tiny oil drop (Millikan expt)	1.5 µm	5×10^{-5}	10^{-7}	bv

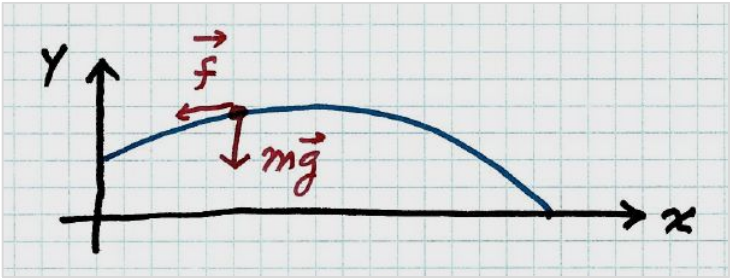
2.2 Linear air resistance

Now we'll specialize to *c* = 0.
I.e., assume that the force of air resistance on a projectile is

$$\vec{f} = -b\vec{v}$$

Then the equation of motion for the projectile moving through air is

$$m\dot{\vec{v}} = m\vec{g} - b\vec{v}$$



<u>Cartesian components</u>
x = horizontal coordinate;
y = vertical coordinate (*let positive be upward*)

$$m\dot{v}_x = -b\,v_x$$
$$m\dot{v}_y = -mg - bv_y$$

The linear case is very nice, because the x and y coordinates *separate*; so we can solve their equations separately.

Recall from PHY 183, we do the same thing if we *<u>neglect</u>* air resistance:

$$x'' = 0 \quad \text{so} \quad x(t) = x_0 + v_{0x}\,t$$
$$y'' = -g \quad \text{so} \quad y(t) = y_0 + v_{0y}\,t - \tfrac{1}{2}\,g\,t^2$$

But today we are introducing frictional force components; so <u>the trajectory is not a parabola</u>.

Special case:
Horizontal motion with linear drag

Figure 2.3

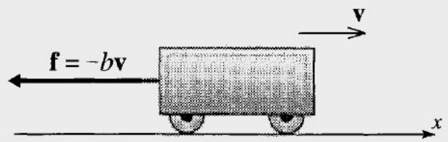


Figure 2.3 A cart moves on a horizontal frictionless track in a medium that produces a linear drag force.

$$m\dot{v}_x = -b\,v_x$$

The solution is obvious ("separate and integrate")

$$v_x(t) = C\,e^{-bt/m}$$

where C is a constant.

Determine C from the initial conditions

$$v_x(t) = v_{0x}\,e^{-bt/m}$$

(or from some other information.)

Figure 2.4

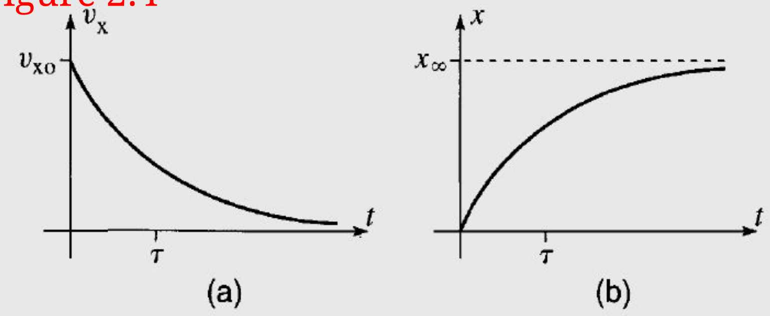


Figure 2.4 (**a**) The velocity v_x as a function of time, t, for a cart moving horizontally with a linear resistive force. As $t \to \infty$, v_x approaches zero exponentially. (**b**) The position x as a function of t for the same cart. As $t \to \infty$, $x \to x_\infty = v_{xo}\tau$.

$$dx = v_x\,dt \quad \leftarrow \text{i.e., } dx' = v_x(t')\,dt'$$

$$\int_{x_0}^{x} dx' = \int_0^t v_x(t')\,dt'$$

$$x - x_0 = v_{0x}\left(\frac{-m}{b}\right) e^{-bt'/m}\Big]_{t'=0}^{t}$$

$$= \frac{m v_{0x}}{b}\left\{1 - e^{-bt/m}\right\}$$

Special case:
Vertical motion with linear drag

We want to solve this equation (*):

$$m\dot{v}_y = +mg - bv_y$$

The solution may be obtained in several ways ...

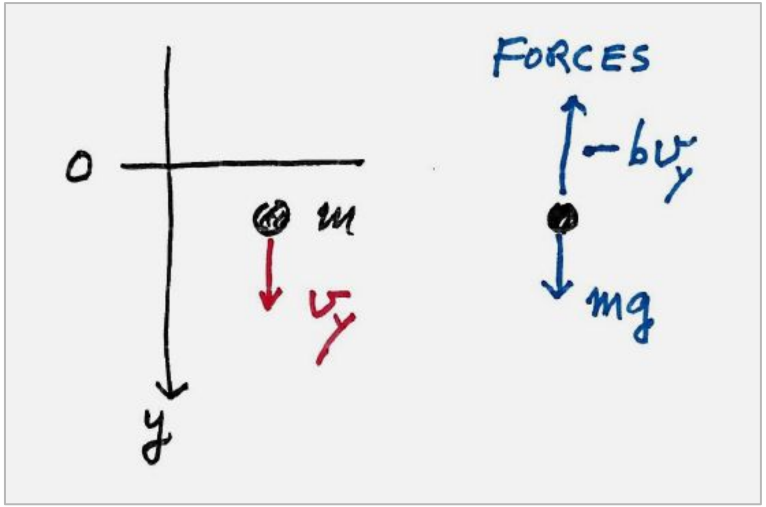
- *trial and error; also called, guessing ;*
- *separation of variables (often used);*
- *particular + homogeneous;*

  the third method only works for linear equations; MTH 235

I'm following Taylor:

Let the y axis point *downward*;

so $(F_y)_{\text{gravity}} = + mg.$



Solution of differential equations by separation of variables

- Suppose we have an equation of this form,

$$\frac{df}{dx} = K(f(x)) \qquad (1)$$

K is a function of f ;

the unknown is f(x) .

- Separate the variables x and f,

$$\frac{df}{K(f)} = dx \quad \leftarrow \text{i.e.,} \quad \frac{df'}{K(f')} = dx' \qquad (2)$$

- Now integrate both sides of the equation,

$$\int_{f_0}^{f} \frac{df'}{K(f')} = \int_{x_0}^{x} dx' = x - x_0 \qquad (3)$$

Eq.(3) gives x as a function of f.

- But what we want is f as a function of x. So finally use algebra to solve (3) for f,

f(x) = the solution of (3) (4)

Go back to vertical motion with linear drag.

We want to solve this equation(*):

$$m\frac{dv}{dt} = mg - bv$$

Separate:

$$m\,dv = (mg - bv)\,dt$$

$$\frac{dv}{mg-bv} = \frac{dt}{m}$$

Integrate:

$$\int_{v_0}^{v}\frac{dv'}{mg-bv'} = \int_0^t \frac{dt'}{m}$$

$$LHS = \frac{-1}{b}\ln(mg-bv')\Big]_{v'=v_0}^{v} = \frac{1}{b}\ln\frac{mg-bv_0}{mg-bv}$$

$$RHS = t/m$$

Solve:

$$\frac{mg-bv_0}{mg-bv} = e^{bt/m}$$

$$mg-bv_0 = (mg-bv)\,e^{bt/m}$$

$$bv - bv_0 e^{-bt/m} = mg(1-e^{-bt/m})$$

$$v = \left(v_0 - \frac{mg}{b}\right)e^{-bt/m} + \frac{mg}{b}$$

$$v(t) = \left(v_0 - \frac{mg}{b}\right)e^{-bt/m} + \frac{mg}{b}$$

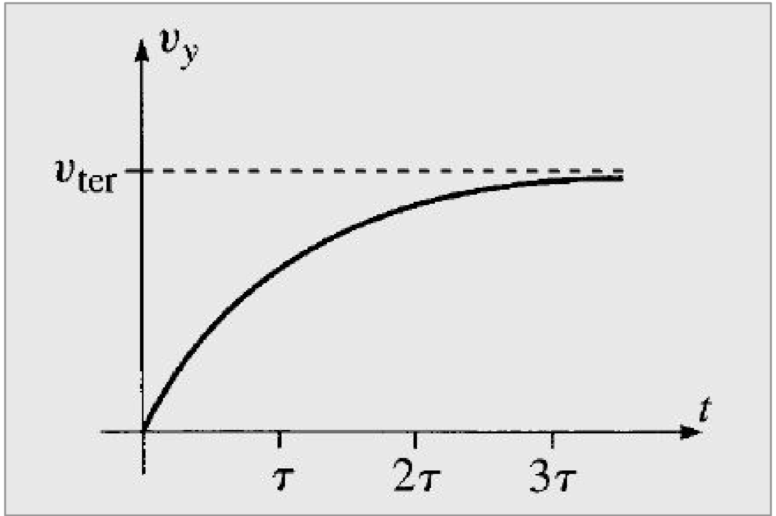
Check Diff. Eq.; Check initial value;

$$m\frac{dv}{dt} + bv = -b\left(v_0 - \frac{mg}{b}\right)e^{-bt/m} + bv = mg \checkmark$$

An object falling in Earth's gravity with a linear aerodynamic drag force, $\boldsymbol{f} = -b\,\boldsymbol{v}$.

$$v(t) = \left(v_0 - \frac{mg}{b}\right) e^{-bt/m} + \frac{mg}{b}$$

Figure 2.6 *(assumes $v_0 = 0$)*



"Terminal velocity" and "time constant"

$$v_{ter} = \lim_{t \to \infty} v(t) = \frac{mg}{b}$$

$$\tau = \frac{m}{b} \qquad \left(e^{-bt/m} = e^{-t/\tau} \right)$$

Also,

determine y(t) by integrating $v_y(t)$.

Example 2.2

TERMINAL SPEED OF A SMALL DROP OF WATER

The terminal velocity of a drop of water (diameter = D) is the velocity at which

$$F = mg - bv - cv^2 = 0.$$

The parameter values for air at STP are

$$b = (1.6 \times 10^{-4} \text{ Ns/m}^2) D$$

$$c = (0.25 \text{ Ns}^2/\text{m}^4) D^2 ;$$

also, $m = (0.52 \times 10^6 \text{ kg/m}^3) D^3$.

Calculate $v_{terminal}$ as a function of D.

Result

Small droplets (e.g., in a cloud) have small $v_{term.}$;
large droplets (e.g., raindrops) have larger $v_{term.}$.

Homework Assignment #3
due in class Wednesday, Sept. 22

[11] Problem 2.2
[12] Problem 2.3
[13] Problem 2.10
[14] Problem 2.18
[15] Problem 2.26
[16] Water drops
[17] Parametric Plot

Use the cover sheet.