Section 2.3 Trajectory and range of a projectile in a linear medium

Recall the basic equations for projectile motion with linear air resistance.

Let $x =$ the horizontal coordinate; let $y =$ the vertical coordinate, with the positive direction upward;

\[ m \ddot{x} = -b \dot{x} \implies \dot{x}(t) = \dot{x}_0 e^{-b t / m} \]

\[ m \ddot{y} = -m g - b \dot{y} \implies \dot{y}(t) = (\dot{y}_0 + \frac{m g}{b}) e^{-b t / m} - \frac{m g}{b} \]

Rewrite the solutions ...

\[ \dot{x}(t) = \dot{x}_0 e^{-t / \tau} \quad \text{where} \quad \tau = \frac{m}{b} \]

\[ \dot{y}(t) = (\dot{y}_0 + \dot{v}_{\text{ter}}) e^{-t / \tau} - \dot{v}_{\text{ter}} \quad \text{where} \quad \dot{v}_{\text{ter}} = \frac{m g}{b} = \frac{g \tau}{b} \]

"time constant" = $\tau$ and
"terminal speed" = $v_{\text{ter}}$
Calculation of the trajectory

The "trajectory" is the curve in space along which the particle moves.

▶ Horizontal position, x(t)

\[ \dot{v}_{x}(t) = v_{0x} e^{-\frac{t}{\tau}} \]
\[ x(t) = \int_{0}^{t} \dot{v}_{x}(t') \, dt' = v_{0x}(-\tau) e^{-\frac{t}{\tau}} \bigg|_{t=0}^{t} = v_{0x} \tau (1 - e^{-\frac{t}{\tau}}) \]

Note: \( x(t) \) approaches a maximum distance, \( x_{\text{max}} = v_{0x} \tau \) as \( t \) increases.

▶ Vertical position, y(t); take \( y(0) = 0 \)

\[ \dot{v}_{y}(t) = (v_{0y} + v_{\text{ter}}) e^{-\frac{t}{\tau}} - v_{\text{ter}} \]
\[ y(t) = \int_{0}^{t} \dot{v}_{y}(t) \, dt' = (v_{0y} + v_{\text{ter}})(-\tau) e^{-\frac{t}{\tau}} \bigg|_{0}^{t} - v_{\text{ter}} t \]
\[ = (v_{0y} + v_{\text{ter}}) \tau \left[ 1 - e^{-\frac{t}{\tau}} \right] - v_{\text{ter}} t \]

Note: \( v_{y}(t) \) approaches a maximum speed – the terminal speed – \( v_{\text{ter}} = \frac{mg}{b} \)

▶ Trajectory, i.e., the curve in space along which the projectile moves; solve for \( y \) as a function of \( x \);

\[ y = \left( \frac{v_{0y} + v_{\text{ter}}}{v_{0x}} \right) \frac{x}{v_{0x} \tau} + v_{\text{ter}} \tau \ln \left( 1 - \frac{x}{v_{0x} \tau} \right) \]
Figure 2.7
The trajectory of a projectile, assuming linear air resistance

Horizontal range \( R \) and \( R_{\text{vac}} \)

Recall, without air resistance,

\[ R_{\text{vac}} = \frac{2v_{0x}v_{0y}}{g} \]

With linear air resistance,

\[ y = 0 \text{ where } x = R \]

\[ \Rightarrow (v_{0y} + v_{\text{fer}}) \frac{R}{v_{0x}} + v_{\text{fer}} \ln \left(1 - \frac{R}{v_{0x} \tau}\right) = 0 \]

But this is a transcendental equation for the range \( R \) (as a function of \( v_{0x}, v_{0y} \) and \( \tau \)).

Methods to compute:
[i] use a computer, or
[ii] make an approximation.
Example 2.4

THE RANGE OF A SMALL METAL PELLET
(note: "small" means tiny).

Suppose $D = 0.2$ mm and $\{ v_{0x}, v_{0y} \} = \{ 1/\sqrt{2}, 1/\sqrt{2} \}$ m/s. Calculate the range.

To solve:

$y = 0 \text{ where } x = R$

$\therefore (v_{0y} + v_{\text{ter}}) \frac{R}{v_{0x}} + v_{\text{ter}} \ln \left( 1 - \frac{R}{v_{0x} \tau} \right) = 0$

Recall, $v_{\text{ter}} = mg/b$ and $\tau = m/b = v_{\text{ter}}/g$.

We'll calculate $m$ from the density $\rho$,

$m = (\pi/6) \rho D^3$;

also, recall for air $b = (1.6 \times 10^{-4} \text{ N.s/m}^2) D$.

♦ If there is no air resistance then the range is

$R_{\text{vac}} = 2 v_{0x} v_{0y}/g = 10.2 \text{ cm}$.

♦ Assuming linear air resistance, and solving the equation exactly (I used Mathematica)

<table>
<thead>
<tr>
<th>material</th>
<th>density (kg m$^{-3}$)</th>
<th>$v_{\text{ter}}$ (m/s)</th>
<th>$R$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gold</td>
<td>$16 \times 10^3$</td>
<td>20.5</td>
<td>9.74</td>
</tr>
<tr>
<td>aluminum</td>
<td>$2.7 \times 10^3$</td>
<td>3.47</td>
<td>7.96</td>
</tr>
</tbody>
</table>

Program: $\{ \rho, D \} \rightarrow m$

$\{ m, b \} \rightarrow \tau$ and $\{ \tau, g \} \rightarrow v_{\text{ter}}$

$\{ v_{0x}, v_{0y}, v_{\text{ter}}, \tau \} \rightarrow R$ (numerically)
Taylor's approximate method ...

\[ y = 0 \quad \text{where} \quad x = R \]
\[ \therefore \left( \frac{v_{oy}}{v_{0x}} + \frac{R}{v_{0x} \tau} \right) + \frac{v_{ter} \ln \left( 1 - \frac{R}{v_{0x} \tau} \right)}{v_{ter}} = 0 \]

We might anticipate that \( R / (v_{0x} \tau) \) is small; these tiny pellets will have a small range.

Let \( \varepsilon = R / (v_{ex} \tau) \) AND ANTICIPATE \( \varepsilon \ll 1 \).

Recall the Taylor series for \( \ln (1 - \varepsilon) \)

\[ = -\varepsilon - \varepsilon^2 / 2 - \varepsilon^3 / 3 + O(\varepsilon^4) \]

so approximate

\[ \ln (1 - \varepsilon) \approx -\varepsilon - \varepsilon^2 / 2 - \varepsilon^3 / 3 \]

Then after a bit of algebra (exercise: check this)

\[ R \approx R_{\text{vac}} \left\{ 1 - \frac{4}{3} \frac{v_{0y}}{v_{\text{ter}}} \right\} \]  
(eq. 2.44)

where \( R_{\text{vac}} = \frac{v_0^2}{g} \).

Results of the approximation:

gold \quad R \approx 9.73 \text{ cm}  
[exact = 9.74 \Rightarrow \text{error} = -0.1\% ]

aluminum \quad R \approx 7.42 \text{ cm}  
[exact = 7.96 \Rightarrow \text{error} = -7.3\% ]
Range versus initial speed for a projectile with linear air resistance

Assume these parameters:
initial angle of elevation = 45 degrees;  terminal speed = 40 m/s.

Black curve = exact, calculated with Mathematica;
Red curve = Taylor's approximation;
Blue dashed curve = result with no air resistance (range =  \( R_{vac} = \frac{v_0^2}{g} \))
A plotting technique:

**PARAMETRIC PLOTS**

Suppose we know \( x(t) \) and \( y(t) \); and now we want to make a plot of \( y \) versus \( x \) ("the trajectory").

\[
\text{ParametricPlot}[ \{x[t], y[t]\}, \{t, 0, 5\}, \text{PlotRange}\rightarrow\{(0,15), (0,6)\} ]
\]

For example, with \( g = 9.8 \text{ m/s}^2 \), \( v_0 = \{10,10\} \text{ m/s} \), and \( \tau = 3 \text{ s} \):

\[
x[t_] := v_0 x \tau (1 - \text{Exp}[-t/\tau])
\]

\[
y[t_] := (v_0 y + v_{ter}) \tau (1 - \text{Exp}[-t/\tau]) - \text{v}_{ter} t
\]

Homework Assignment #3
due in class Friday, September 22

[11] Problem 2.2 *
[12] Problem 2.3 *
[13] Problem 2.10 **
[14] Problem 2.18 *
[15] Problem 2.26 *
[16] Water drops
[17] Parametric plot

*Use the cover sheet.*