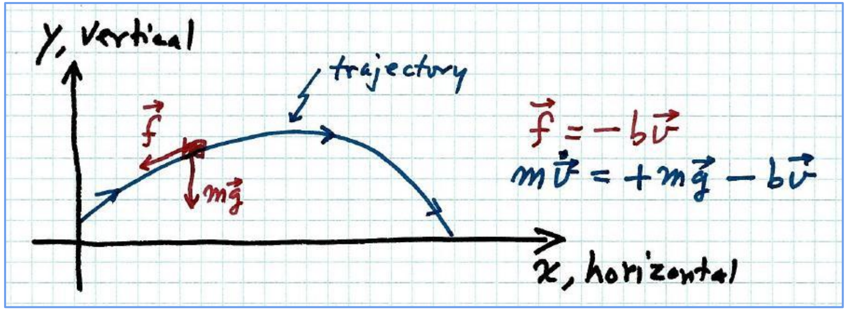
Section 2.3 *Trajectory and range of a projectile in a linear medium*

Recall the basic equations for projectile motion with linear air resistance.

Let x = the horizontal coordinate;
let y = the vertical coordinate, with the positive direction upward;



Solutions for v_x and v_y as functions of t …

$$m\dot{v}_x = -b v_x \quad \Rightarrow \quad v_x(t) = v_{0x}\, e^{-bt/m}$$

$$m\dot{v}_y = -mg - b v_y$$

$$\Rightarrow \quad v_y(t) = \left(v_{0y} + \frac{mg}{b}\right) e^{-bt/m} - \frac{mg}{b}$$

Rewrite the solutions …

$$v_x(t) = v_{0x}\, e^{-t/\tau} \qquad \text{where } \underline{\tau = \frac{m}{b}}$$

$$v_y(t) = \left(v_{0y} + v_{ter}\right) e^{-t/\tau} - v_{ter}$$

$$\text{where } \underline{v_{ter} = \frac{mg}{b}} = \tau g$$

"time constant" = τ and

"terminal speed" = v_{ter}

Calculation of the trajectory

The "trajectory" is the curve in space along which the particle moves.

▸Horizontal position, x(t)

$$v_x(t) = v_{0x}\, e^{-t/\tau}$$

$$x(t) = \int_0^t v_x(t')\, dt'$$

$$= v_{0x}(-\tau)\, e^{-t'/\tau}\Big]_{t'=0}^{t}$$

$$= v_{0x}\,\tau\,(1 - e^{-t/\tau})$$

Note: x(t) approaches a maximum distance, $x_{max} = v_{0x}\,\tau$ as t increases.

▸Vertical position, y(t); **take y(0) = 0**

$$v_y(t) = (v_{0y} + v_{ter})\, e^{-t/\tau} - v_{ter}$$

$$y(t) = \int_0^t v_y(t')\, dt'$$

$$= (v_{0y} + v_{ter})(-\tau)\, e^{-t'/\tau}\Big]_0^t - v_{ter}\, t$$

$$= (v_{0y} + v_{ter})\,\tau\,[1 - e^{-t/\tau}] - v_{ter}\, t$$

Note: $v_y(t)$ approaches a maximum speed – the terminal speed – v_{ter} = mg /b

▸Trajectory, i.e., the curve in space along which the projectile moves;
solve for y as a function of x ;

$$y = (v_{0y} + v_{ter})\,\frac{x}{v_{0x}} + v_{ter}\,\tau\,\ln\left(1 - \frac{x}{v_{0x}\tau}\right)$$

Figure 2.7

The trajectory of a projectile, assuming linear air resistance

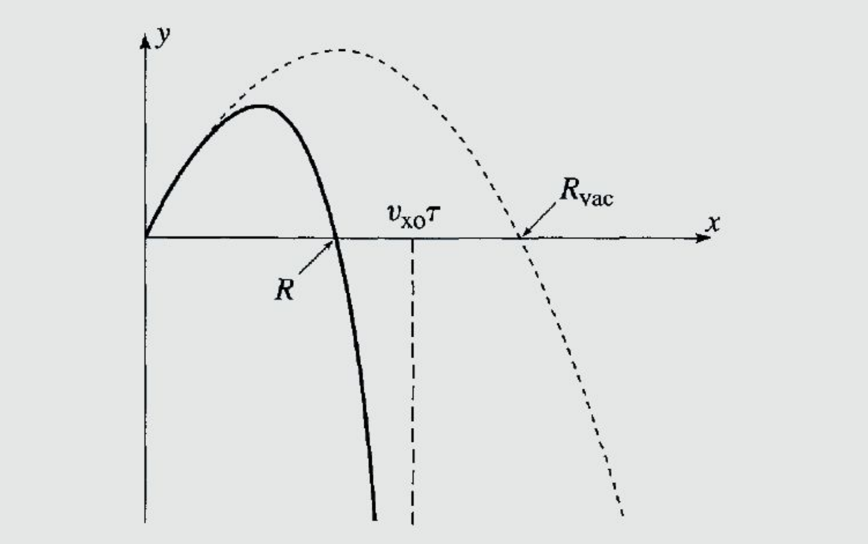


Figure 2.7 The trajectory of a projectile subject to a linear drag force (solid curve) and the corresponding trajectory in a vacuum (dashed curve). At first the two curves are very similar, but as *t*

Horizontal range ; R and R_{vac}

Recall, without air resistance,

$$R_{vac} = \frac{2 v_{0x} v_{0y}}{g}$$

With linear air resistance,

$y = 0$ where $x = R$

$$\therefore \left(v_{0y} + v_{ter}\right)\frac{R}{v_{0x}} + v_{ter}\,\tau \ln\left(1 - \frac{R}{v_{0x}\tau}\right) = 0$$

But this is a *transcendental equation* for the range R ***(as a function of v_{0x} , v_{0y} and τ).***

Methods to compute:
[i] use a computer, or
[ii] make an approximation.

Example 2.4

THE RANGE OF A SMALL METAL PELLET

(note: "small" means tiny).

Suppose D = 0.2 mm and { v_{0x} ,v_{0y} } = { 1/√2 , 1/√2 } m/s. *Calculate the range.*

To solve:

$y = 0$ where $x = R$

$$\therefore \left(v_{0y} + v_{ter}\right)\frac{R}{v_{0x}} + v_{ter}\,\tau \ln\left(1 - \frac{R}{v_{0x}\tau}\right) = 0$$

Recall, v_{ter} = mg/b and τ = m/b = v_{ter}/g.
We'll calculate m from the density ρ,

$$m = (\pi/6)\,\rho\, D^3 \;;$$

also, recall for air b = (1.6 x 10^{-4} N.s/m^2) D .

♦ If there is no air resistance then the range is

$$R_{vac} = 2\, v_{0x}\, v_{oy}/g = 10.2 \text{ cm} .$$

♦ Assuming linear air resistance, and solving the equation exactly (I used *Mathematica*)

material	density (kg m^{-3})	v_{ter} (m/s)	R (cm)
gold	16 ×10^3	20.5	9.74
aluminum	2.7 ×10^3	3.47	7.96

Program: {ρ , D} ⟶ m
{m , b}⟶ τ and {τ , g} ⟶ $v_{term.}$
{ v_{0x} , v_{0y} , $v_{term.}$, τ } ⟶ R (*numerically*)

♦ Taylor's approximate method ...

$y = 0$ where $x = R$

$$\therefore \; (v_{0y} + v_{ter}) \frac{R}{v_{0x}} + v_{ter}\,\tau \ln\left(1 - \frac{R}{v_{0x}\tau}\right) = 0$$

We might anticipate that $R / (v_{0x}\,\tau)$ is small ; these tiny pellets will have a small range.

Let $\varepsilon = R / (v_{ex}\,\tau)$ **AND ANTICIPATE $\varepsilon << 1$.**

Recall the Taylor series for $\ln(1 - \varepsilon)$

$$= -\varepsilon - \varepsilon^2/2 - \varepsilon^3/3 + O(\varepsilon^4)\;;$$

so approximate

$$\ln(1 - \varepsilon) \approx -\varepsilon - \varepsilon^2/2 - \varepsilon^3/3$$

Then after a bit of algebra

(exercise: check this)

$$R \approx R_{vac} \left\{ 1 - \frac{4\;v_{0y}}{3\;v_{ter}} \right\} \qquad \text{(eq. 2.44)}$$

where $R_{vac} = v_0^{\,2}/g$.

Results of the approximation:

gold $R \approx 9.73$ cm

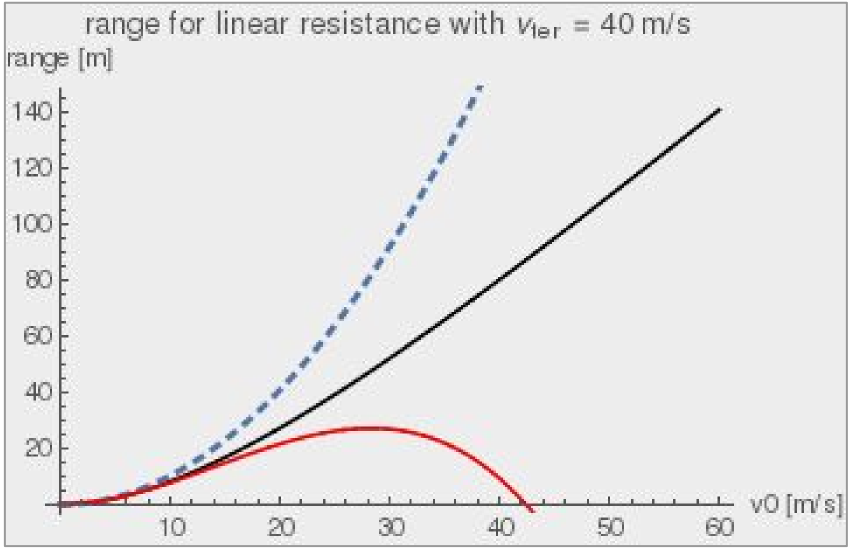
[exact = 9.74 ⇒ error =−0.1 %]

aluminum $R \approx 7.42$ cm

[exact = 7.96 ⇒ error =−7.3 %]

Range versus initial speed for a projectile with linear air resistance

Assume these parameters:
initial angle of elevation = 45 degrees;   terminal speed = 40 m/s.



Black curve = exact, calculated with Mathematica;

Red curve = Taylor's approximation;

Blue dashed curve = result with no air resistance (range = $R_{vac} = v_0^{\,2}/g$)

A plotting technique:

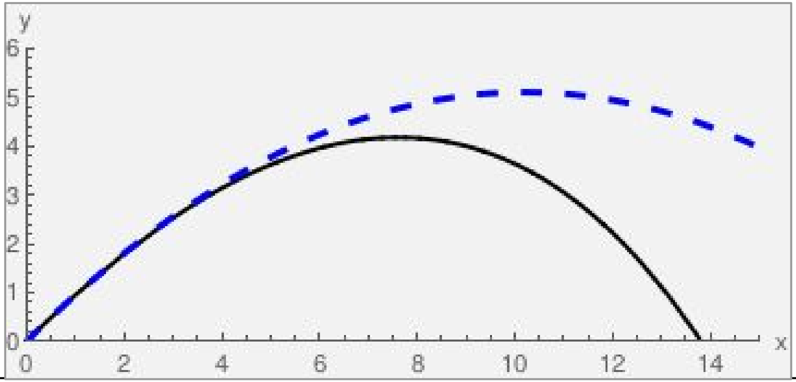
PARAMETRIC PLOTS

Suppose we know *x(t)* and *y(t)* ;

and now we want to make a plot of *y* versus *x* (*" the trajectory "*) .

```
ParametricPlot[ {x[t],y[t]},
 {t,0,5},
 PlotRange->{{0,15},{0,6}}  ]
```

For example, with g = 9.8 m/s^2 ,
v_0 = {10,10} m/s , and τ = 3 s :

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x[t_]:=v0x*τ*(1 - Exp[-t/τ])

y[t_]:=(v0y+vter)*τ*(1 - Exp[-t/τ])
            - vter*t
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Homework Assignment #3
due in class Friday, September 22
[11] Problem 2.2 *
[12] Problem 2.3 *
[13] Problem 2.10 **
[14] Problem 2.18 *
[15] Problem 2.26 *
[16] Water drops
[17] Parametric plot
Use the cover sheet.