Section 2.4

Quadratic Air Resistance

Read Section 2.4.

Now we'll assume that the air resistance force is proportional to the *square* of the speed; i.e., the linear resistance is negligible compared to the quadratic resistance.

Equations

- $\boldsymbol{f} = -\, c\, v^2\, \boldsymbol{e}_v$
- For a spherical object, with diameter D,

  $c = \gamma\, D^2$, $\gamma = 0.25\ \mathrm{Ns^2\, m^{-4}}$

Horizontal motion with quadratic air resistance; e.g., *a bicycle coasting on a level road;*

$$m \frac{dv}{dt} = -c v^2$$

Solve by separation of variables

$$m\, dv = -c v^2\, dt$$

$$\frac{dv}{v^2} = -\frac{c}{m}\, dt \quad \leftarrow \text{i.e.,} \quad \frac{dv'}{v'^2} = -\frac{c}{m}\, dt'$$

Integrate:

$$\int_{v_0}^{v} \frac{dv'}{v'^2} = -\frac{c}{m} \int_0^t dt' \rightarrow = -\frac{c}{m} t$$

$$\hookrightarrow = -\frac{1}{v'} \Big]_{v_0}^{v} = \frac{-1}{v} + \frac{1}{v_0}$$

Solve for v:

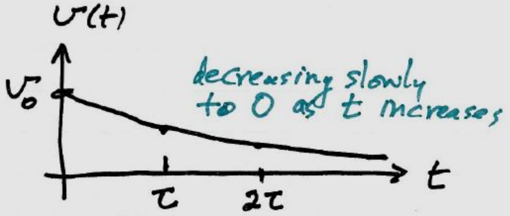
$$\frac{1}{v} = \frac{1}{v_0} + \frac{ct}{m} \quad \text{OR} \quad v(t) = \frac{m v_0}{m + c v_0 t}$$

Figure 2.8.

Result

$$v(t) = \frac{v_0}{1 + t/\tau} \quad \text{where} \quad \tau = \frac{m}{c v_0}$$

time constant for quadratic resistance

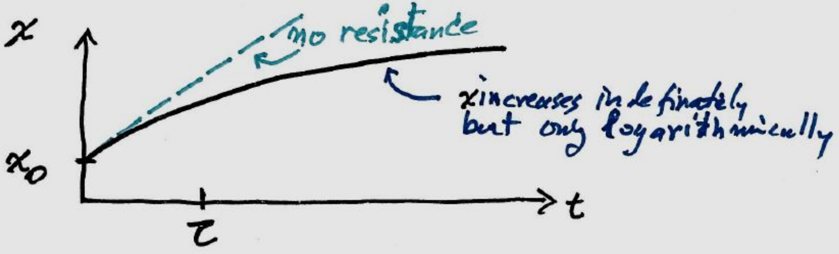


The position $x(t)$

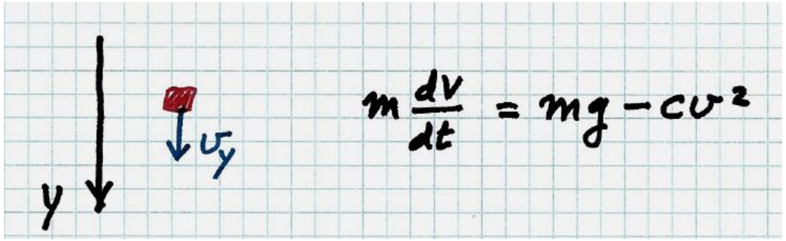
$$\int_{x_0}^{x} dx' = \int_0^t \underbrace{v_x(t')}_{\frac{v_0}{1 + t/\tau}} dt'$$

$$x - x_0 = v_0 \tau \ln(1 + t'/\tau)\Big]_{t'=0}^{t}$$

$$= v_0 \tau \ln(1 + t/\tau)$$



<u>Vertical motion with quadratic air resistance;</u> e.g., an object dropped from rest falls straight down; ($v_0 = 0$);



It is convenient to let the y axis point downward, so $F_g = + mg$.

Before we solve the equation, *what is the terminal velocity?*

At terminal velocity, $F_{net} = 0$.

Thus $v_{ter} = \sqrt{mg/c}$.

Separation of variables:

$$m\,dv = (mg - cv^2)\,dt$$

$$\frac{dv'}{mg - cv'^2} = \frac{dt'}{m} \qquad v_{term.} = \sqrt{\frac{mg}{c}}$$

Integration:

$$\int_{v_0}^{v} \frac{dv'}{mg - cv'^2} = \frac{1}{c}\int_{v_0}^{v} \frac{dv'}{v_{term.}^2 - v'^2}$$

$$= \frac{1}{c\,v_{term.}} \operatorname{arctanh}\left(\frac{v'}{v_{term.}}\right)\Big|_{v_0}^{v}$$

$$= \frac{1}{c\,v_{term}} \operatorname{arctanh}\left(\frac{v}{v_{term.}}\right) \quad \text{for } v_0 = 0$$

$$= \int_0^t \frac{dt'}{m} = t/m$$

Solve for $v(t)$:

$$\operatorname{arctanh}\left(\frac{v}{v_{term}}\right) = \frac{c\,v_{term.}\,t}{m} = \frac{gt}{v_{term}}$$

$$v(t) = v_{ter.} \tanh\left(\frac{gt}{v_{ter.}}\right)$$

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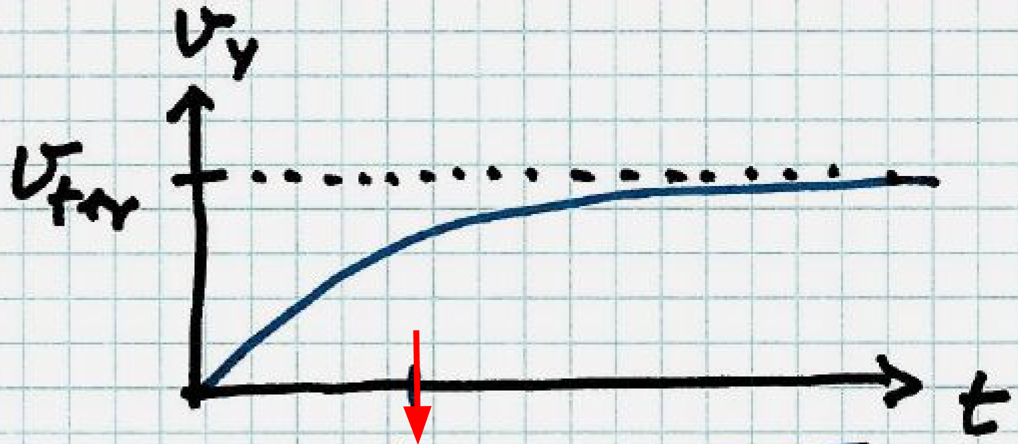

$$\frac{v_{ter}}{g} = \sqrt{\frac{m}{gc}}$$

Example 2.5

a baseball dropped from a high tower

(See Figures 2.9 (a) and (b) .)

Dashed : no air resistance

Black : quadratic air resistance

Mathematica commands

Recall, $c = \gamma D^2$

```
{mass, diam} = {0.15 , 0.07} (* kg and m *)
{g, gamma} = {9.8,0.25} (* mks units *)
vterm = Sqrt[mass*g/(gamma*diam^2)]
vy[t_] := vterm*Tanh[g*t/vterm]
y[t_] := vterm^2/g*Log[Cosh[g*t/vterm]]
p1 = Plot[{g*t, vy[t]}, {t, 0, 8}]
p2 = Plot[{1/2*g*t^2, y[t]}, {t, 0, 8}]
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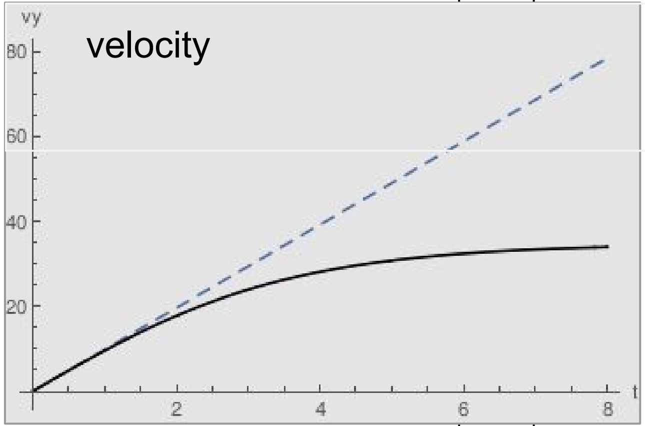
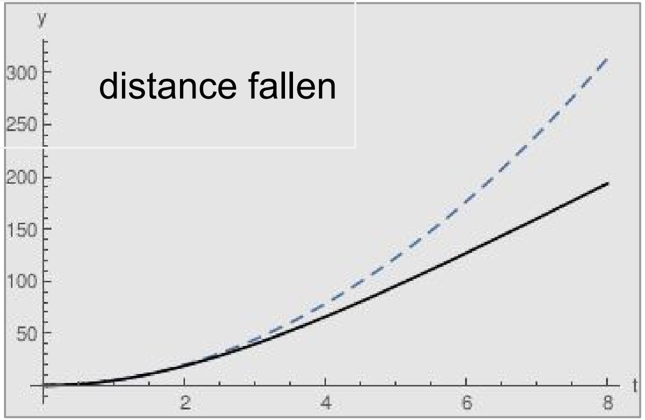


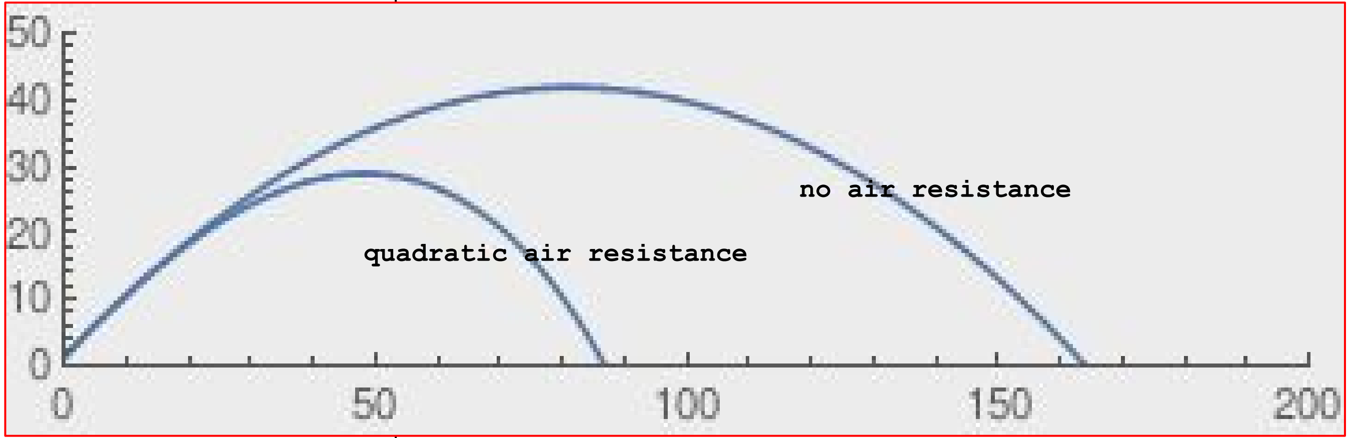


Quadratic drag for a projectile with both x (horizontal) and y (vertical) motion

Example. A baseball home run, calculated using Mathematica.

Assume these *initial conditions*:
speed = 40 m/s (~ 90 mi /h)
angle = 45 degrees
height = 1 m

y [meters]



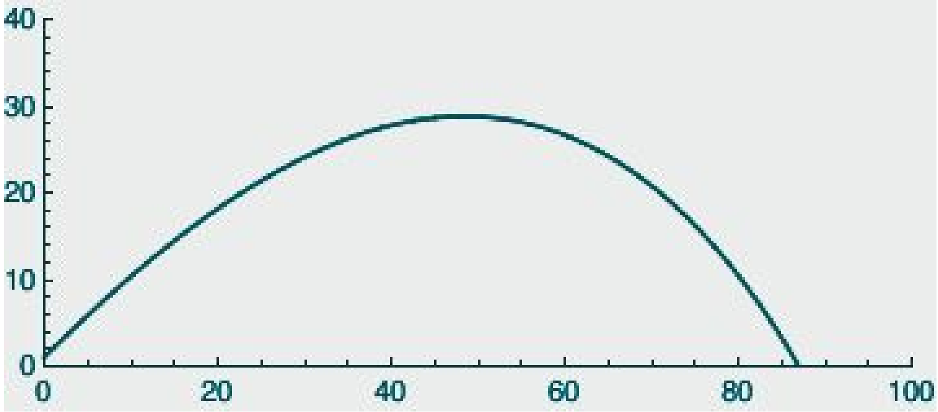
x [meters]

Mathematica Program

```
g = 9.8;  vter = 35;     m = 0.15; const = m*g/vter^2;
v0 = 40;  ang0=Pi/4;
eqs = {
   m*x''[t] ==    - const*x'[t]*Sqrt[x'[t]^2 + y'[t]^2],
   m*y''[t] ==  -m*g - const*y'[t]*Sqrt[x'[t]^2 + y'[t]^2],
   x[0] == 0, x'[0] == v0*Cos[ang0],
   y[0] == 1, y'[0] == v0*Sin[ang0]   };
Q = NDSolve[eqs, {x[t], y[t]}, {t, 0, 6}];
ParametricPlot[{x[t] /. Q[[1]], y[t] /. Q[[1]]}, {t, 0, 6}]
```

$= -c\,(v_x^2 + v_y^2)\cos\theta$

$= -mg - c\,(v_x^2 + v_y^2)\sin\theta$



Test yourself:

A bicycle rider coasts *down a hill*.

The angle of the slope is θ = 10 degrees = 0.174 radians. (A) Using your knowledge about air resistance, *estimate* the terminal speed of the bicycle, in meters per second. (B) Determine the speed as a function of time, starting from speed v_0 at t = 0.

Data:

Mass = 70 kg ;

effective area = 1 m^2 .

Homework Assignment #3

due date = Wednesday Sept. 20

[11] Problem 2.2 *

[12] Problem 2.3 *

[13] Problem 2.10 **

[14] Problem 2.18 *

[15] Problem 2.26 *

[16] Water drops

[17] Parametric Plot

Use the cover sheet.