Section 2.5.

Motion of a Charged Particle in a Magnetic Field

The magnetic force on a charged particle is the *Lorentz force*,

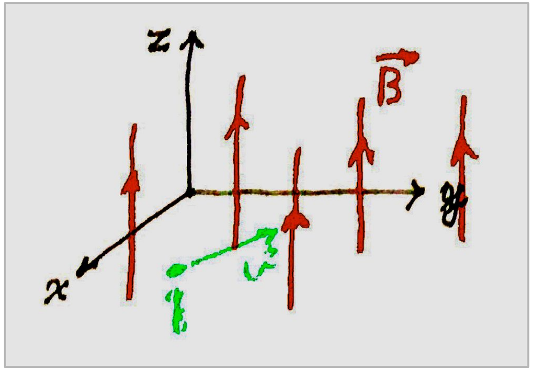
$$\mathbf{F} = q\,\boldsymbol{v} \times \mathbf{B}\,. \qquad (1)$$

Here **B** is the magnetic field. (PHY 184)

In general, **B** = **B(r**,t) ;
in Eq. (1) **B** means the field *at the position of the charged particle.*

We'll keep it simple, and assume that **B** is uniform in space and constant in time.

Figure 2.12



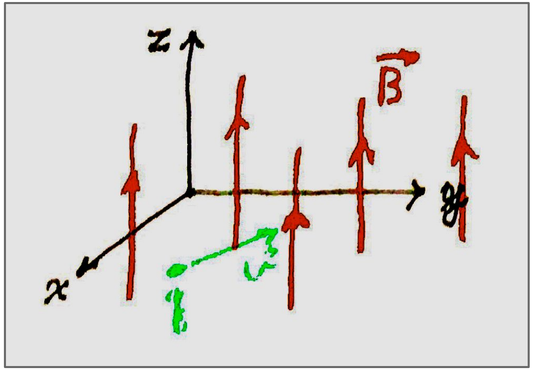
Charge q moves with velocity $\boldsymbol{v}$ in a magnetic field **B.**
Calculate the trajectory.
The goal is to solve this equation of motion,

$$m\,\frac{d\mathbf{v}}{dt} = q\,\mathbf{v} \times \mathbf{B}$$

Cartesian coordinates

Assume B is uniform and constant.

Set up a coordinates system such that the z axis is in the direction of B.



The equations of motion

$$m \frac{d\mathbf{v}}{dt} = q\,\mathbf{v} \times \mathbf{B}$$

$$m\,\boldsymbol{v} = m\,\{ v_x , v_y , v_z \}$$

$$m\,\dot{\boldsymbol{v}} = m\,\{ \dot{v}_x , \dot{v}_y , \dot{v}_z \}$$

(dot means d/dt)

$$q\,\mathbf{v} \times \mathbf{B} = q \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = q\,\{ \hat{e}_x v_y B - \hat{e}_y v_x B \}$$

$$q\,\mathbf{v} \times \mathbf{B} = q\,\{ v_y B , -v_x B , 0 \}$$

Solutions

The z component

$$m \dot{v}_z = 0$$

$$v_z = v_{0z}, \text{ constant}$$

$$z(t) = z_0 + v_{0z} t$$

The transverse components

$$m \dot{v}_x = q B v_y$$

$$m \dot{v}_y = -q B v_x$$

The cyclotron frequency

$$\dot{v}_x = \frac{qB}{m} v_y = \omega v_y$$

where $\omega = qB/m$

$$\ddot{v}_x = \omega \dot{v}_y = -\omega^2 v_x$$

$$v_x = c_1 \cos \omega t + c_2 \sin \omega t$$

$$v_y = \frac{1}{\omega} \dot{v}_x = -c_1 \sin \omega t + c_2 \cos \omega t$$

$\therefore$ $\vec{v}$ sweeps out a circle of radius $\sqrt{c_1^2 + c_2^2}$.

EXERCISE: $|\mathbf{v}|$ is constant .

Results

v_z = constant ; $z = v_{0z} t$

$$v_x = c_1 \cos\omega t + c_2 \sin\omega t$$

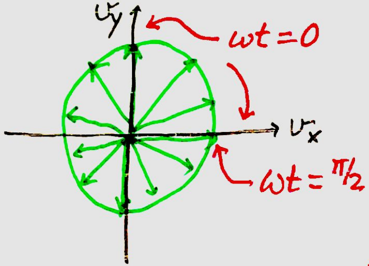
$$\Rightarrow x = \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t$$

$$v_y = -c_1 \sin\omega t + c_2 \cos\omega t$$

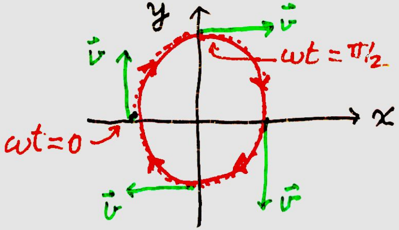
$$\Rightarrow y = \frac{c_1}{\omega}\cos\omega t + \frac{c_2}{\omega}\sin\omega t$$

Assume $v_x(0) = 0$; then $c_1 = 0$.

$$\vec{v}(t) = c_2 \left\{ \hat{e}_x \sin\omega t + \hat{e}_y \cos\omega t \right\}$$



The speed is constant;
$\sqrt{v_x^2 + v_y^2} = c_2$



The trajectory is a circle;
radius = c_2/ω ;
direction = CLOCKWISE

$$\vec{r}(t) = x(t)\,\hat{e}_x + y(t)\,\hat{e}_y$$

The period is $2\pi/\omega$.
The frequency is $\omega/(2\pi)$.
ω is called the *angular frequency*.

It is interesting to analyze the problem using ***complex numbers***.

Define

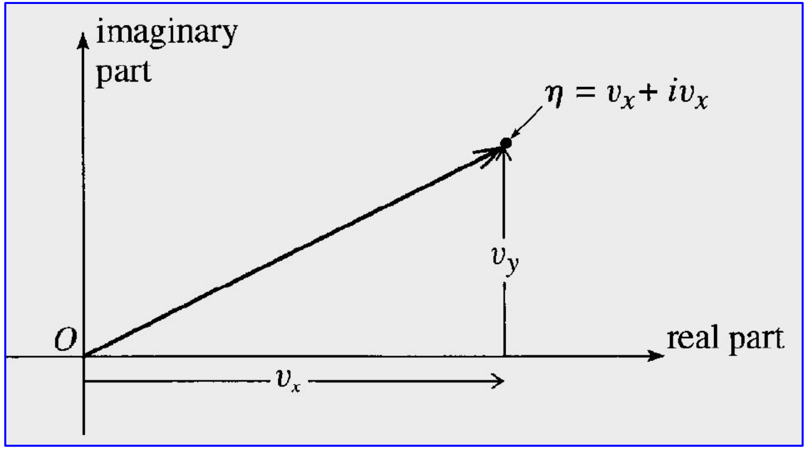
$$\eta = v_x + i\, v_y$$

$$i = \sqrt{(-1)}$$

That is,

$$v_x = \mathrm{Re}\, \eta$$
$$v_y = \mathrm{Im}\, \eta$$

Figure 2.13 :
The plane of complex numbers



Now write the equations of motion (transverse components) in terms of η.

Solve the equations of motion using the complex variable

$$\eta = v_x + i v_y$$

$$\dot{\eta} = \dot{v}_x + i \dot{v}_y$$

$$= \omega v_y + i(-\omega) v_x$$

$$= -i\omega (v_x + i v_y) = -i\omega\eta$$

$$\dot{\eta} = -i\omega \eta$$

The exponential function

$$\frac{df}{dx} = \alpha f(x) \Rightarrow f(x) = A e^{\alpha x}$$

because then

$$\frac{df}{dx} = A\alpha e^{\alpha x} = \alpha f(x) \checkmark$$

So $\eta(t)$ is an exponential function

$$\eta(t) = A e^{-i\omega t} \quad \text{Complex!}$$

In the next section [Section 2.6] we'll review some important properties of complex numbers and the complex exponential function —– *widely useful in theoretical physics*.

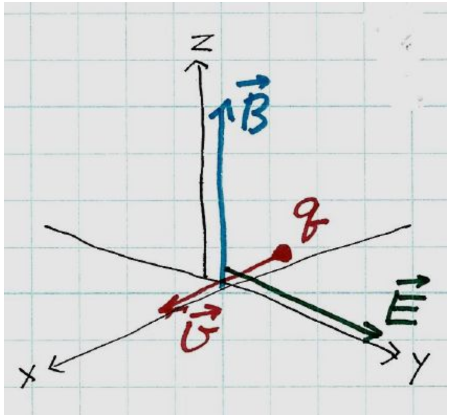
Aside Some related problems:

A charge q moving in both magnetic and electric fields,

$$\mathbf{F} = q\,(\,\mathbf{E} + \boldsymbol{v} \times \mathbf{B}\,)$$

- If **E** and **B** are parallel :
  Taylor Problem 2.53 (***easy***)

- If **E** and **B** are perpendicular :



- If $\boldsymbol{v} = \mathbf{E} \times \mathbf{B}/B^2$ then the charge moves through the fields with constant velocity.

Proof

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

$$= q\vec{E} + q\,\frac{(\vec{E}\times\vec{B})}{B^2} \times \vec{B}$$

• $(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{A}(\vec{B}\cdot\vec{C})$

$$= q\vec{E} + \frac{q}{B^2}\Big[\vec{B}\underbrace{(\vec{E}\cdot\vec{B})}_{\substack{0 \\ (\perp\ \text{fields})}} - \vec{E}\underbrace{(\vec{B}\cdot\vec{B})}_{B^2}\Big] = 0$$

QED by Newton's first law

- In the most general case, the charge has a "drift velocity" **E x B** / B^2 ; PHY 481 ; in general the trajectory is a cycloid curve.

Homework Assignment #4
due in class Wednesday, September 27

[17] Problem 2.23 *
[18] Problem 2.31 **
[19] Problem 2.41 **
[20] Problem 2.53 *
[21] Problem 2.43 *** [computer]
[22] Graph $f_n(x)$.

Use the cover sheet.