Section 2.6

Complex Exponentials

Read Section 2.6.

Today's subject isn't classical mechanics.

It's useful mathematics that can be applied in many fields.

So we are going off on a tangent today, to learn about the subject of ...

Complex Exponentials

The exponential function, *exp*(x)

What is this function?
I.e., how is it defined?

I'm sure that you know

$$exp(x) = e^x$$

where e is a certain number.

But what is e?

e is an irrational number, approximately equal to 2.718.

But what is the *exact* value of e?
And what is so special about 2.718…?

The exponential function, *exp*(x)

What is this function?

The definition of *exp*(x) is that

$$exp'(x) = exp(x) \,,$$

with $exp(0) = 1$.

We'll express *exp*(x) as a power series.

But first we need …

Taylor's theorem

If $f(x)$ is continuous and differentiable then

$$f(x+\delta) = f(x) + f'(x)\,\delta + f''(x)\,\delta^2/2 + f'''(x)\,\delta^3/6 + \ldots + f^{(n)}(x)\,\delta^n/n! + \ldots$$

Proof

Compare LHS and RHS as functions of δ

- set $\delta = 0$: $f(x) = f(x)$ check
- differentiate ***w.r.t. δ*** and set $\delta = 0$:

  $f'(x) = f'(x)$ check
- twice differentiate **"** and set $\delta = 0$:

  $f''(x) = f''(x)$ check
- n times differentiate **"** and set $\delta = 0$:

  $f^{(n)}(x) = f^{(n)}(x)$ check

Q.E.D.

The power series for *exp*(u)

Apply Taylor's theorem,
with x = 0 and δ = u .

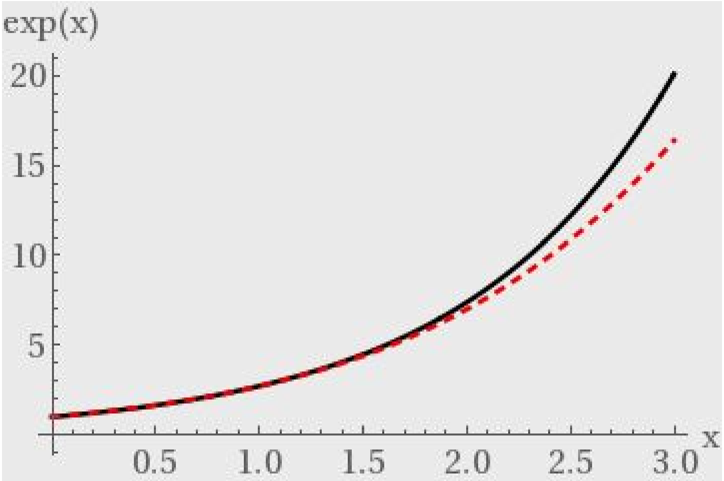
$$exp(u) = exp(0) + exp'(0)\ u$$
$$+ exp''(0)\ u^2/2$$
$$+ exp'''(0)\ u^3/6$$
$$+ \ldots + exp^{(n)}(0)\ u^n/n! + \ldots$$

∴ by the definition of *exp*(x)

$$exp(u) = 1 + u + u^2/2 + u^3/6$$
$$+ \ldots + u^n/n! + \ldots$$

Result

$$exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



red dashes :
truncation of the series at n=4

<u>Theorem</u> *exp*(x) *exp*(y) = *exp*(x+y)

<u>Proof</u>

(This is Taylor's Problem 2.51.)

$$\exp(x+y) = \sum_{p=0}^{\infty} (x+y)^p / p!$$

$$= \sum_{p=0}^{\infty} \underbrace{\sum_{n=0}^{p} x^n y^{p-n} \binom{p}{n}}_{\text{binomial theorem}} \frac{1}{p!}$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{p} \sum_{m=0}^{p} \delta_K(m, p-n)\, x^n y^m \frac{p!}{n!\,m!} \frac{1}{p!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^m}{n!\,m!} \qquad \left(\sum_p = 1\right)$$

$$= \exp(x)\, \exp(y)$$

<u>Corollary</u> *exp*(x) = e^x

where e is a certain number.

<u>Proof</u> : $e^x e^y = e^{x+y}$

e , *the base of natural logarithms*

e may be expressed in terms of the integers, using e = *exp*(1) = e^1 = e :

By the power series,

e = 1 /0! + 1 /1! +1 /2! + 1 /3! + 1 /4! + …

e ≈ 1 + 1 + 0.5 + 0.1667 + 0.0416 + …

e ≈ 2.718

e = 2.718281828459045090 8 …

e is the sum of reciprocal factorials .

Now consider

$$e^{i\theta}$$

where θ is a real number.

It's defined by the power series, so

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

The terms with n odd are imaginary, and the terms with n even are real,

$i^0 = 1$	$n\|4$	(divisible by 4)
$i^1 = i$	$(n-1)\|4$	
$i^2 = -1$	$(n-2)\|4$	
$i^3 = -i$	$(n-3)\|4$	
$i^4 = 1$	$n\|4$	

So,

$$e^{i\theta} = \sum_{n \text{ even}} \frac{(-1)^{n/2}}{n!}\theta^n + i \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n!}\theta^n$$

$$\mathrm{Re} = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots$$

$$= \cos\theta$$

$$\mathrm{Im} = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots$$

$$= \sin\theta$$

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Magnitude and Phase of a complex number, z

Let z denote a complex number.

We can write $z = x + i\,y$;

$x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$.

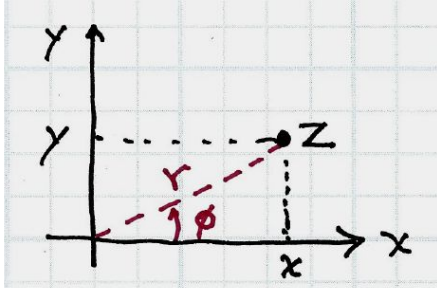
The *magnitude* of z is *r* defined by

$$z^*z = r^2 = (x-iy)\,(x+iy) = x^2 + y^2 \;;$$

the *phase* of z is φ, defined by

$$x = r \cos\varphi \,,$$
$$y = r \sin\varphi \,;$$

$$\text{or,}\quad \tan\varphi = y/x \,.$$



By Euler's formula,

$$z = r\, e^{i\varphi} \quad .$$

This is a crucial trick when we use complex numbers in theoretical physics.

We can write $z = x + i\,y$;

or we can write $z = r\, e^{i\,\varphi}$.

We just use whichever representation is more convenient.

In the next lecture we'll use the complex exponential function to calculate the motion of a charged particle in a magnetic field.

<u>Homework Assignment #4</u>

due in class Wednesday 9/27

[17] Problem 2.23 *

[18] Problem 2.31 **

[19] Problem 2.41 **

[20] Problem 2.53 *

[21] Problem 2.43 *** [computer]

[22] Graph of $f_n(x)$

Use the cover sheet.