Section 3. ***Rockets***

Read Section 3.2.

The rocket contains a fuel that burns rapidly. As the exhaust gas is expelled from the combustion chamber, there is a reaction force on the rocket.

"THRUST"

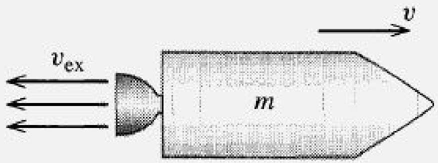


Figure 3.2 A rocket of mass m travels to the right with speed v and ejects spent fuel with exhaust speed v_{ex} relative to the rocket.

Now we'll analyze the motion, using the principle of momentum; i.e.,

$$\dot{\mathbf{P}} = \mathbf{F}^{ext} \quad . \qquad \textit{eq. (3.1)}$$

But be careful to identify **P** correctly!

(Dot • means d/dt)

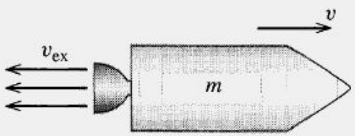


Figure 3.2 A rocket of mass m travels to the right with speed v and ejects spent fuel with exhaust speed v_{ex} relative to the rocket.

<u>Derive the equation of motion for the rocket.</u>

What I mean by the "rocket" is the metal cylinder plus the fuel <u>*inside*</u>.

The mass of the rocket decreases as fuel is expelled out the back.

Let

m(t) = the mass of the rocket at time t ;

$$m(t) = M_{cyl} + M_{fuel\ inside}(t) \quad .$$

The equation of motion will depend on *two parameters* of the rocket engine:

- v_{ex} = the ***relative speed*** of the exhaust gas.

The velocity of the exhaust gas at time t is $v(t) - v_{ex}$, where v(t) is the velocity of the rocket.

- ***K*** = the ***mass rate*** of the exhaust; *K is positive.*

$$K = - dM_F / dt = - dm/dt$$

I.e., K δt = the mass of fuel expelled by the engine during the time δt
= the *decrease* of the rocket mass during δt.
(units of K = kg/s)

Derivation

Consider the change of momentum of the total system = ***rocket and fuel*** , from time t to time $t + \delta t$.

The total momentum at time t is

$$P(t) = m(t)\, v(t) + P_{\text{fuel already expelled before t}}$$

(I'm only considering one-dimensional motion of the rocket.)

The total momentum at time $t + \delta t$ is

$$P(t+\delta t) = m(t+\delta t)\, v(t+\delta t)$$ **←rocket at t+dt**

$$+ (K\,\delta t)\,[\, v(t) - v_{ex} \,]$$ **←fuel expelled during dt**

$$+ P_{\text{fuel already expelled before t}}$$

Subtle point here : because δt is small (in fact, take the limit $\delta t \to 0$) we can neglect the change of v during the time δt when we calculate the velocity of the exhaust.

The *change* of total momentum =

$$\delta P = P(t+\delta t) - P(t) .$$

The equation of motion is Eq. 3.1,

$$dP/dt = \delta P/\delta t = F_{ext} .$$

Now calculate ...

$$\delta P = (\, m(t) - K\,\delta t \,)\,(\, v(t) + \delta v \,) + K\,\delta t\,(v(t) - v_{ex}) + P_{\text{expelled before t}} - P(t)$$

$$= K\,\delta t\,(\, - v_{ex} \,) + m(t)\,\delta v + O(\delta^2)$$

(• note the cancellations!; • neglect order δ^2)

Result: $m\,(dv/dt) - K\,v_{ex} = F_{ext}$

The rocket equations

For a rocket moving in one dimension, the velocity v(t) obeys

$$m \frac{dv}{dt} = K v_{ex} + F_{ext} \qquad (1)$$

where v_{ex} = the relative speed of the exhaust and K = the mass rate of the exhaust. Here m(t) is the mass of the rocket *including enclosed fuel*, so

$$\frac{dm}{dt} = -K \qquad (2)$$

Thrust force = $K v_{ex}$.

Example. A rocket in deep space with constant values of v_{ex} and K ...

Let the initial velocity be v_0 and the initial mass be m_0.

Calculate the velocity at time t.

By Eq.(2), $m(t) = m_0 - Kt$

Then Eq.(1) is

$$(m_0 - Kt)\frac{dv}{dt} = K v_{ex}$$

- Separation of variables:

$$dv = K v_{ex} \frac{dt}{m_0 - Kt}$$

- Integration of both sides of the eq.:

$$v - v_0 = K v_{ex}\left(\frac{-1}{K}\right) \ln(m_0 - Kt')\Big]_{t'=0}^{t'=t}$$

$$= -v_{ex}\{\ln(m_0 - Kt) - \ln(m_0)\}$$

$$v = v_0 + v_{ex} \ln[\, m_0 / (m_0 - K t)]$$

Example #2. A rocket in deep space with constant v_{ex} and arbitrary time dependence of K(t) .

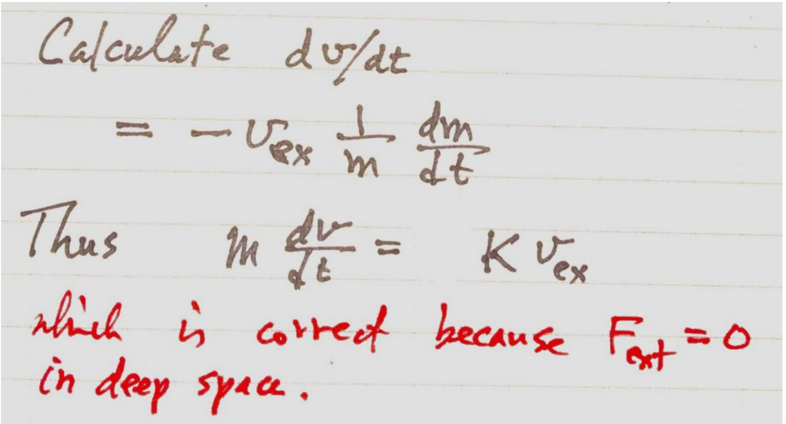
Let the initial velocity be v_0 and the initial mass be m_0.

Calculate the velocity at time t.

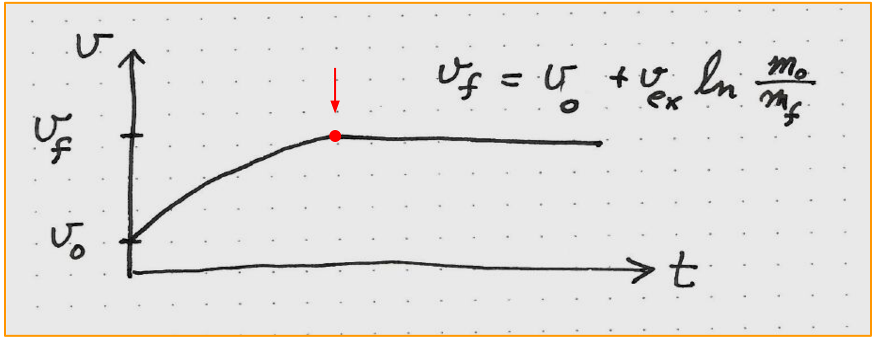
Result:

$$v(t) = v_0 + v_{ex}\, ln\, [m_0\ /m(t)\ ]$$

PROOF



GRAPH OF VELOCITY VS TIME



$$\mathcal{F}_{ext} = 0 \Rightarrow$$

$$\delta v = v_{exhaust}\, ln\, (\, m_0 / m\,)$$

TAKE OFF FROM EARTH'S SURFACE

Here we have

$$m\,(dv/dt) = K\,v_{ex} - m\,g$$

Assume

g = constant and v_{ex} = constant.

Now solve the DIFF. EQ.

$$m\,dv = K\,v_{ex}\,dt - mg\,dt$$

can't integrate ! don't know m(t).

$$= -\,v_{ex}\,dm - m\,g\,dt$$

$$dv = -\,v_{ex}\,(dm\,/m) - g\,dt$$

$$dv' = -\,v_{ex}\,(dm'\,/m') - g\,dt'$$

Integrate

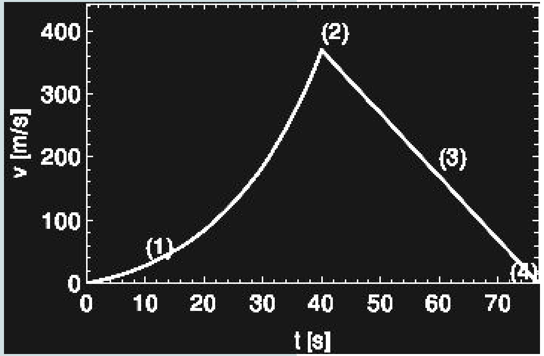
$$\int_0^v dv' = -\,v_{ex}\int_{m0}^{m} dm'/m' - g\int_0^t dt'$$

$$v(t) = v_{ex}\,\ln\,[m_0\,/m(t)] - g\,t\ .$$

Also, $m(t) = m_0 - K\,t$ *(assuming K is constant)*

Graph of v as a function of t

(1) initial slope = $(v_{ex}K - m_0g)/m_0$
(2) rocket runs out of fuel
(3) final slope = $-\,g$
(4) rocket starts to fall downward
(5) Max. height = area under the curve



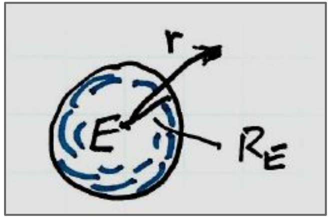
A more difficult example:

Fire a rocket to the stars

Now g is not constant. As r increases, g decreases;

$$g(r) = g_s \, R_E^{\,2} / r^2$$

for $r > R_E$.



The equation of motion is

$$m \frac{dv}{dt} = K v_{ex} - m g_s \frac{R_E^2}{r^2}$$

Example: Saturn V rocket

<u>*1st stage:*</u> burn time = 165 sec;
fuel = 2,003,000 *kg* ; thrust = 35,000 *kN*

<u>*2nd stage:*</u> burn time = 360 sec;
fuel = 456,000 *kg* ; thrust = 5,000 *kN*

<u>*3rd stage:*</u> burn time = 500 sec;
fuel = 110,000 *kg* ; thrust = 1,000 *kN*

QUIZ

Suppose a rocket takes off from Earth's surface.
Parameters

burn time = 165 sec ;
m0 = 2,500,000 kg ; fuel = 2,000,000 *kg* ;
thrust = 35,000 *kN* .

Calculate the velocity at the burn-out time.

Hint: Three steps:
K = ______________________________
v_exhaust = ______________________
v_final = __________________________