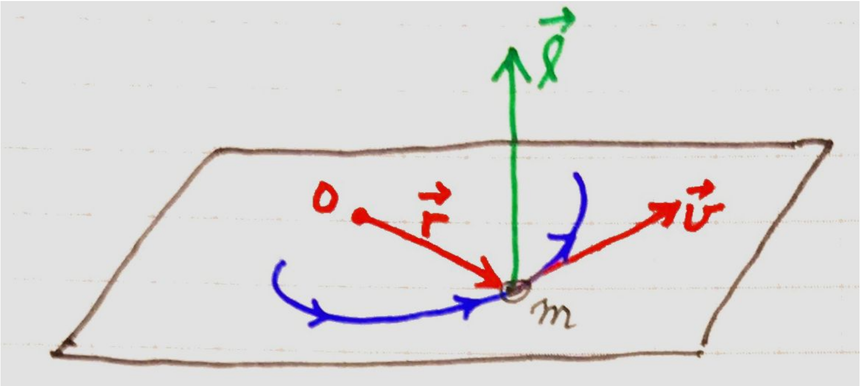
Section 3.4.
Angular Momentum for a Single Particle

<u>The definition</u>

Consider a particle with position vector $\boldsymbol{r}$ (w.r.t. the chosen origin O) and momentum $\boldsymbol{p}$ = m $\boldsymbol{v}$.



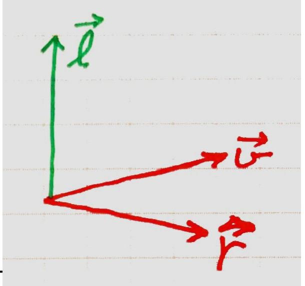
Define the angular momentum, of the particle about the origin O;

- notation = $\boldsymbol{\ell}$;
- $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$
- $\boldsymbol{\ell}$ is a vector.

(Review cross product; page 7 and the right hand rule)

- The direction is perpendicular to the plane spanned by $\boldsymbol{r}$ and $\boldsymbol{p}$.

- The magnitude is r p sin θ .



Theorem.

$d\boldsymbol{\ell}/dt$ is equal to the torque.

Proof.

$$\frac{d\vec{\ell}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \text{ where } \vec{p} = m\vec{v}$$

$$= m\left[\underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{v \times \vec{v} = 0} + \vec{r} \times \underbrace{\frac{d\vec{v}}{dt}}_{\frac{d\vec{v}}{dt} = \vec{F}/m}\right]$$

$$= \vec{r} \times \vec{F} = \text{torque}$$

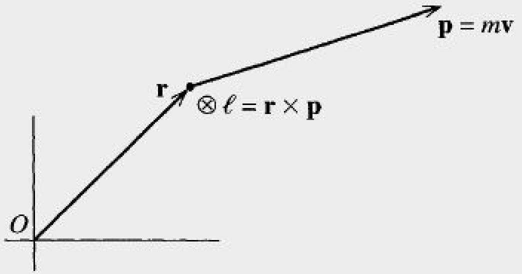


Figure 3.5 For any particle with position $\mathbf{r}$ relative to the origin O and momentum $\mathbf{p}$, the angular momentum about O is defined as the vector $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$. For the case shown, $\boldsymbol{\ell}$ points into the page.

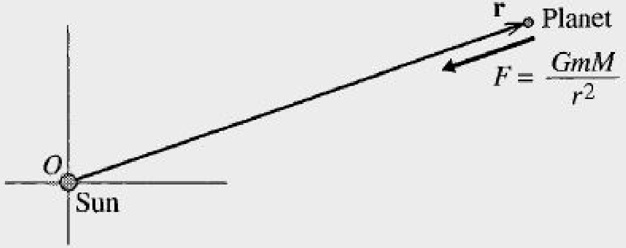


Figure 3.6 A planet (mass m) is subject to the central force of the sun (mass M). If we choose the origin at the sun, then $\mathbf{r} \times \mathbf{F} = 0$, and the planet's angular momentum about O is constant.

Example

Kepler's second law

Figure 3.7

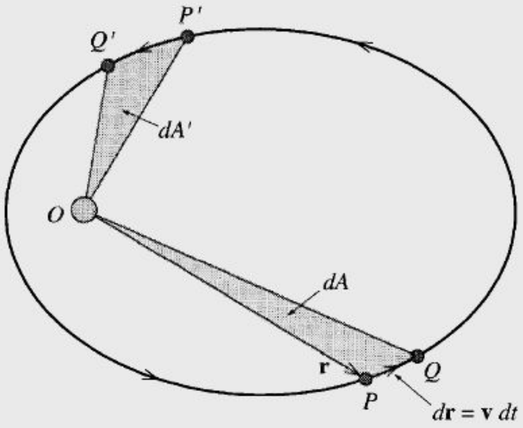


Figure 3.7 The orbit of a planet with the sun fixed at O. Kepler's second law asserts that if the two pairs of points P, Q and P', Q' are separated by equal time intervals, $dt = dt'$, then the two areas dA and dA' are equal.

Johannes Kepler published three laws of planetary orbits, in 1609 and 1619.

He determined these laws from a mathematical analysis of planetary observations −− very difficult in the 17th century ; Kepler was a mathematical genius.

Kepler's second law:

> *The radial vector sweeps out equal areas in equal times.*

We'll prove that this follows from *conservation of angular momentum.*

Angular momentum was not known at the time of Kepler; it was Newton who discovered that ***any*** radial force implies the law of equal areas.

Comment:

"Kepler's second law" —

or, conservation of angular momentum —

applies to ***all*** central forces.

I.e., it's not just for planetary orbits.

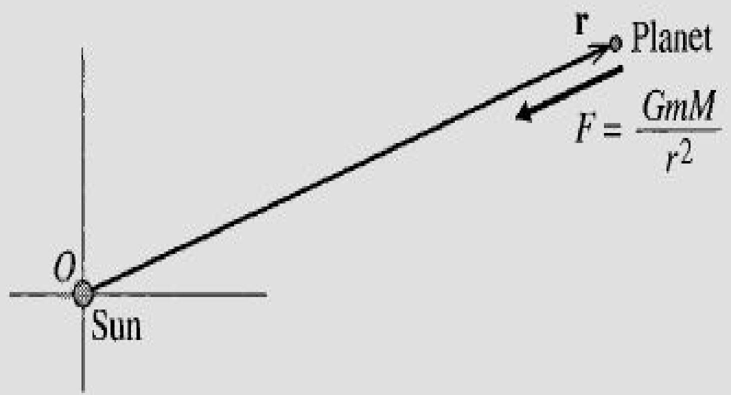
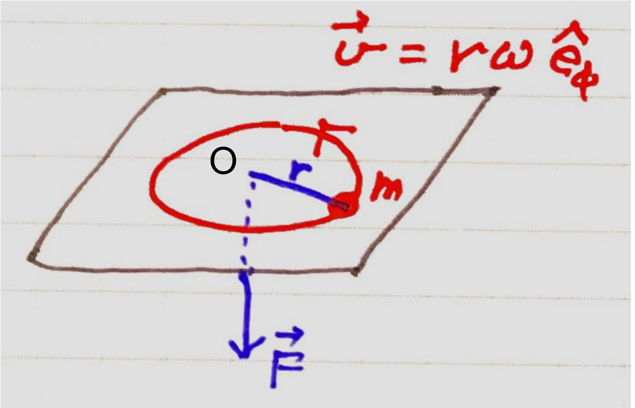


Figure 3.6 A planet (mass m) is subject to the central force of the sun (mass M). If we choose the origin at the sun, then $\mathbf{r} \times \mathbf{F} = 0$, and the planet's angular momentum about O is constant.

TAYLOR PROBLEM 3.25.



The mass *m* slides without friction on a horizontal surface. It is attached to a string as shown. The string goes through a hole in the surface, *O*; and it can be pulled down beneath the surface to change the distance *r* from *m* to *O*.

A. Initially, $r = r_0$ and $\omega = \omega_0$. Calculate F_0 required to keep r constant.

B. Then the string is pulled down by distance $r_0/2$. Determine the final angular velocity.

C. Calculate the work done pulling the string.

(A)

$$F_0 = m\, r_0\, \omega_0^2$$

(B) Ang. momentum is constant, so

$$\ell = m\, r^2\, \omega$$
$$= m\, r_0^2\, \omega_0$$

$$\therefore \quad \omega_{final} = 4\, \omega_0$$

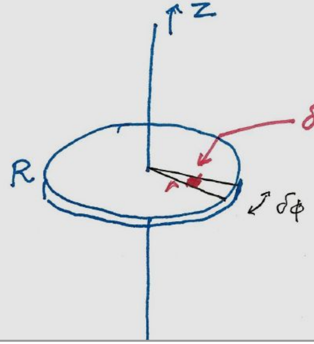
(C)

$$W = \int F\, dy$$

should be ΔK

Example

Calculate the angular momentum of a rotating disk, that rotates around the symmetry axis of the disk.



For uniform density,

$$\frac{\delta m}{M} = \frac{\delta A}{A} = \frac{\delta r\,(r\,\delta\phi)}{\pi R^2}$$

$$dm = \frac{M}{\pi R^2}\, r\,dr\,d\phi$$

$$\vec{L} = \sum_{\delta m} \vec{r} \times \delta m\, \vec{v} \quad \text{where} \quad \vec{r} = r\,\hat{e}_r, \quad \vec{v} = r\omega\,\hat{e}_\phi$$

$$\left[v = \frac{2\pi r}{T} = \omega r\right]$$

$$\vec{L} = \int dm\; r^2 \omega\; \hat{e}_z$$

$$\vec{L} = I\omega\,\hat{e}_z \quad \text{where} \quad I = \int dm\; r^2$$

$$I_{DISK} = \frac{M}{\pi R^2}\int_0^R r\,dr \int_0^{2\pi} d\phi\; r^2$$

$$= \frac{M}{\pi R^2}\,\frac{R^4}{4}\,2\pi = \frac{1}{2}MR^2$$

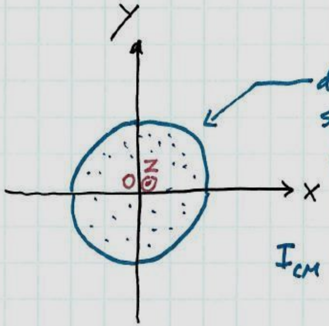
Moment of Inertia

With respect to axis z,

$$I = \int dm\; r^2 = \int \rho\, dV\; r^2$$

$$\vec{L} = I\omega\,\hat{e}_z$$

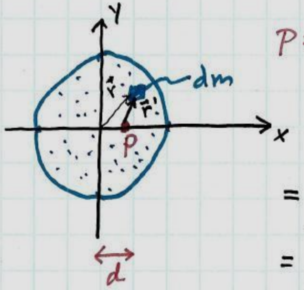
THE PARALLEL AXIS THEOREM – an example



disk ;
symmetry about z axis
CoM = 0

$$I_{CM} = \int r_\perp^2\, dm = \int (x^2+y^2)\, dm$$

$$= \frac{1}{2} M R^2$$



P = parallel axis at distance d

$$I_p = \int (x'^2 + y'^2)\, dm$$

$$= \int \left[(d-x)^2 + y^2 \right] dm$$

$$= \int \left[d^2 - 2xd + x^2 + y^2 \right] dm$$

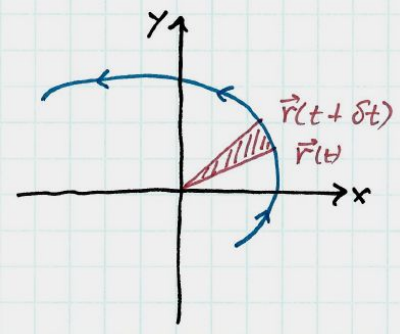
$$= M d^2 + I_{CM} \quad \text{because } \langle x \rangle = 0$$

$$\int x\, dm = 0$$

$$I_p = I_{CM} + M d^2$$

KEPLER'S SECOND LAW …

First, the orbit lies in a plane because the vector $\boldsymbol{\ell}$ is a constant of the motion.



Use plane polar coordinates (r,φ). The area swept out by the radial vector from time t to t+δt is

$$\delta A = \frac{1}{2}\, r \; r\, \delta\phi$$

$$\frac{\delta A}{\delta t} = \frac{1}{2}\, r^2\, \frac{\delta \phi}{\delta t}$$

The angular momentum is

$$\vec{\ell} = \vec{r} \times m\vec{v}$$

$$\vec{r} = r(t)\, \hat{e}_r(t)$$

$$\vec{v} = \dot{r}\, \hat{e}_r + r\, \dot{\hat{e}}_r$$

$$= \dot{r}\, \hat{e}_r + r\dot{\phi}\, \hat{e}_\phi$$

$$\vec{\ell} = m\, r\hat{e}_r \times \left(\dot{r}\hat{e}_r + r\dot{\phi}\, \hat{e}_\phi \right)$$

$$= m\, r^2 \dot{\phi}\, \hat{e}_z$$

$$\therefore \quad \frac{dA}{dt} = \frac{\ell}{2m} \quad \text{is constant}$$

The area rate is constant because angular momentum is constant.

Homework Assignment #6
due in class Wednesday, October 11

[27] Problem 3.16 *

[28] Problem 3.20 **

[29] Problem 3.22 **

[30] Problem 3.27 **

[31] Problem 3.32 **

[32] Problem 3.35 **

Use the cover sheet.

The first midterm exam is Friday, October 6.