- Section 4.3: Force as the gradient of potential energy
- Section 4.4: The second condition for a force to be conservative

*The gradient operator* ( $\nabla$, called "del" ) in Cartesian coordinates, is

$$\nabla = \boldsymbol{e}_x\,(\partial/\partial x) + \boldsymbol{e}_y\,(\partial/\partial y) + \boldsymbol{e}_z\,(\partial/\partial z)$$

The Oxford Dictionary of Physics:

***Given a scalar function f and a unit vector n, the scalar product n . ∇f is the rate of change of f in the direction of n.***

## Sec. 4.3. $\mathbf{F} = -\nabla U$

*Theorem.* A conservative force is equal to *the negative gradient* of the corresponding potential energy function.

*Proof*

Recall $\Delta U = -W$

i.e. $U(\vec r) - U(\vec r_0) = -\int_{\vec r_0}^{\vec r} \vec F(\vec r')\cdot d\vec r'$

Now $$\frac{\partial U}{\partial x} = [U(\vec r + \epsilon\hat x) - U(\vec r)]/\epsilon \quad (\text{lim. } \epsilon \to 0)$$

$$= -\int_{\vec r_0}^{\vec r+\epsilon\hat x} + \int_{\vec r_0}^{\vec r} \,]\, \vec F\cdot d\vec r'/\epsilon$$

$$= -\int_{\vec r}^{\vec r+\epsilon\hat x} \frac{\vec F\cdot d\vec r'}{\epsilon} = -F(\vec r)\cdot\hat x$$

So $\frac{\partial U}{\partial x} = -F_x$ ; or, $F_x = -\frac{\partial U}{\partial x}$.

Generalize, $\vec F = -\nabla U$.

## Example 4.4      finding ***F*** from U

- $\boldsymbol{F} = -\nabla U$ ⬅
- U is a scalar; ***F*** is a vector.
- In *Cartesian coordinates*, the gradient of U is the vector of partial derivatives.

Taylor gives this example:

$$U(x,y,z) = Axy^2 + B \sin Cz$$

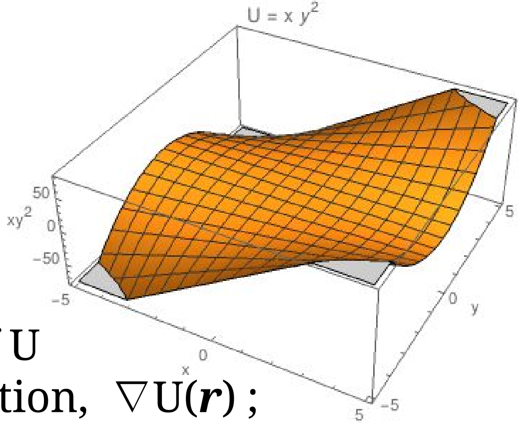
$\therefore \quad F_x = -A y^2 \quad ; \qquad F_y = -2Axy \; ;$

$F_z = -BC \cos Cz$

➡ ***partial derivatives*** ⬅

A 2d example

$U(x,y) = x\, y^2$




The gradient of U is a vector function, $\nabla U(\boldsymbol{r})$ ;

$dU = \nabla U \cdot d\boldsymbol{r}$ ;

Magnitude and Direction of $\nabla U$ :

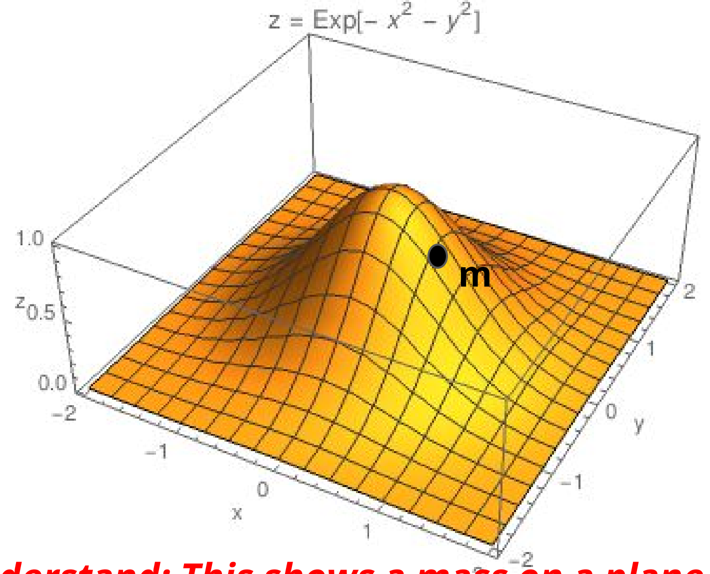
The maximum rate of change of U is $|\nabla U|$, and the direction of the max rate of change is parallel to $\nabla U$. *Note that* $\nabla U$ *points from low U to high U.*

- In *spherical polar coordinates*, see the formulas inside the back cover of the book;

$$= \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

## Here is an example that shows the meaning of the gradient.

Consider a mass m on a plane (xy) , repelled from the origin. The picture shows a surface plot of U(x,y) .




**Understand: This shows a mass on a plane, not a mass on a hill!**

- The potential energy function is
  $U(x,y) = A\ Exp\ [\ -x^2 - y^2\ ]$.
- The gradient of U = U (x,y) is

$$\frac{\partial U}{\partial x} = A\, e^{-(x^2+y^2)}(-2x)$$

$$\nabla U = -2A\, e^{-(x^2+y^2)}\left[x\hat{e}_x + y\hat{e}_y\right]$$

$$= -2A\, e^{-(x^2+y^2)}\,\vec{r}$$

$$= -2A\, e^{-r^2}\,\vec{r}$$

- Direction : $-\hat{\mathbf{r}}$
- Magnitude : $2A\, r \exp(-r^2)$

*The force points radially away from the origin; **F** points from high potential energy toward low potential energy.*

## *SEC. 4.4. THE SECOND CONDITION FOR A FORCE TO BE CONSERVATIVE.*

First, we define another differential operator of vector calculus.

The curl operator , $\boldsymbol{\nabla} \times \boldsymbol{A}$

> Oxford Dictionary of Physics: "Curl: The vector product of the gradient operator with a vector function."

$\boldsymbol{A}$ is a vector, and $\boldsymbol{\nabla} \times \boldsymbol{A}$ is also a vector.

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) \quad \text{[Cartesian]}$$

$$= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right] \quad \text{[spherical polar]}$$

### Theorem

For a conservative force $\boldsymbol{F}$,

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}\,.$$

### Proof:

The curl of a gradient is always 0…

The $\hat{x}$ component of $\nabla \times \vec{A}$ is

$$X = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

Now suppose $\vec{A} = \nabla\alpha$ where $\alpha = \alpha(x,y,z)$.

Then

$$X = \frac{\partial}{\partial y}\frac{\partial\alpha}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\alpha}{\partial y} = \frac{\partial^2\alpha}{\partial y\,\partial z} - \frac{\partial^2\alpha}{\partial z\,\partial y} = 0$$

Since $\boldsymbol{F}$ can be written as a gradient, i.e., $\mathbf{F} = -\nabla U$ ,

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}\,. \qquad \textit{QED}$$

<u>DEFINITION</u> A force $\vec{F}$ is conservative if

(i) $\vec{F}$ depends only on $\vec{r}$

and (ii) $\int_a^b \vec{F} \cdot d\vec{r}$ is independent of the path from a to b.

or (ii′) $\nabla \times \vec{F} = 0$.

<u>THEOREM</u> If $\vec{F}$ is conservative then we can write $\vec{F} = -\nabla U$.

Example 4.5

Is the Coulomb force conservative?

Figure 4.7

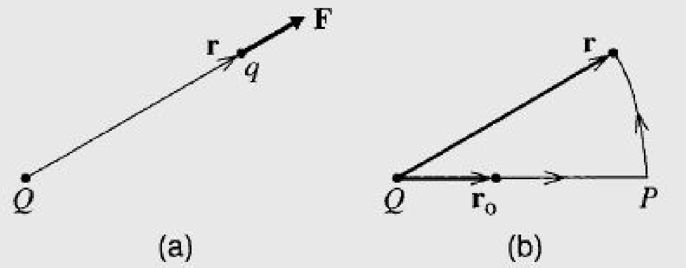



Figure 4.7 (a) The Coulomb force $\mathbf{F} = \gamma \hat{\mathbf{r}} / r^2$ of the fixed charge $Q$ on the charge $q$. (b) The work done by $\mathbf{F}$ as $q$ moves from $\mathbf{r}_o$ to $\mathbf{r}$ can be evaluated following a path that goes radially outward to $P$ and then around a circle to $\mathbf{r}$.

$$\boldsymbol{F(r)} = \hat{\boldsymbol{e}}_{\boldsymbol{r}}\, \gamma / r^2 = \vec{\mathbf{r}}\, \gamma / r^3$$

***What is the potential energy function?***

Assuming ***F*** is conservative,

$$W(\vec{r}_0 \to \vec{r}) = \underbrace{\int_{\vec{r}_0}^{P} \frac{\gamma \hat{e}_x}{r'^2} \cdot \hat{e}_x \, dr'}_{\int_{r_0}^{r} \frac{\gamma}{r'^2} dr'} + \underbrace{\int_{P}^{\vec{r}} \frac{\gamma \hat{r}}{r^2} \cdot \hat{e}_\phi \, r\, d\phi}_{\substack{=0 \\ \hat{r} \perp \hat{e}_\phi}}$$

$$= \left. \frac{-\gamma}{r'} \right]_{r'=r_0}^{r}$$

$$= -\frac{\gamma}{r} + \frac{\gamma}{r_0}$$

$$= -\nabla U = -U(r) + U(r_0)$$

$$U(r) = \gamma / r$$

***∴ The Coulomb force is conservative,***

***and the potential energy is $\gamma / r$ .***

Exercise: Prove $\nabla \times \vec{F} = 0$

(i) in Cartesian coordinates;

(ii) in Polar coordinates.

So, we have two criteria …

(ii) The work done by the force
*(on the object on which the force acts)*
as the object moves from ***a*** to ***b*** is independent of the path from ***a*** to ***b***;

$$W(a \to b) = \int_\Gamma \mathbf{F} . d\boldsymbol{r}$$

$$= U(a) - U(b), \quad \text{indep. of path } \Gamma \,;$$

**or, (ii') The curl of *F* is 0.**

Taylor problem 4.25: "The proof that the condition $\nabla \times \mathbf{F} = 0$ guarantees the path independence of the work $\int_1^2 \mathbf{F}.d\boldsymbol{r}$ done by **F** is unfortunately too lengthy to be included here." ***And then Taylor assigns it as a homework problem!***

**Stokes's theorem**

This is a famous theorem in vector calculus, similar to Gauss's theorem.

Recall Gauss's theorem:

$$\int_V \nabla \cdot \vec{A}\, d^3r = \oint_{S = \text{boundary of } V} \hat{n} \cdot \vec{A}\, dS$$

Stokes's theorem:

$$\int_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS = \oint_{C = \text{boundary of } S} \vec{A} \cdot d\vec{r}$$

Now, suppose $\nabla \times \mathbf{F} = \mathbf{0}$.
Then Stokes's theorem implies $\oint \mathbf{F} . \mathbf{dr} = 0$, around any closed path.
Therefore $\int_a^b \mathbf{F} . \mathbf{dr}$ is path independent, because $a \to b \to a$ is a closed path ;
$(\int_a^b \mathbf{F} . \mathbf{dr})_1 - (\int_a^b \mathbf{F} . \mathbf{dr})_2 = 0.$

Check your understanding:

Use both Cartesian coordinates and spherical polar coordinates. Use the formulas in the back cover of the book.

Prove:

$$\nabla r = \mathbf{e}_{\mathbf{r}}$$

$$\nabla \times \mathbf{r} = \mathbf{0}$$

$$\nabla \times (\mathbf{e}_{\varphi}) = \mathbf{e}_z \;/\; (r \sin\theta)$$

Homework Assignment #7

due in class Wednesday, October 18

[31] Problem 4.3 **

[32] Problem 4.8 **

[33] Problem 4.9 **

[34] Problem 4.10 *

[35] Problem 4.18 **

[36] Problem 4.23 **

[37] the three exam questions (hints are available in the lecture of October 9).

*Use the cover sheet!*