- Section 4.5
  Time-dependent potential energy
- Section 4.6
  Energy for linear 1D motion

§ When the force on a particle depends on time, energy is not conserved.

§ It is not a conservative force.

§ It may still be true that

$$\mathbf{F} = -\nabla U;$$

but U must depend on time;

§ I.e., consider $U = U(\mathbf{r},t)$ and $\mathbf{F} = -\nabla U$.

4.5. Time-dependent potential energy

Suppose $\mathbf{F} = \mathbf{F}(\mathbf{r},t)$, and $\mathbf{F} = -\nabla U(\mathbf{r},t)$.

Here is an example…

Figure 4.8

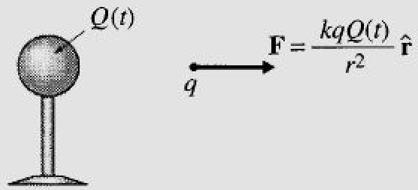


Figure 4.8 The charge $Q(t)$ on the conducting sphere is slowly leaking away, so the force on the small charge q varies with time, even if its position $\mathbf{r}$ is constant.

$$\therefore \quad U(r,t) = k\, q\, Q(t)\, /r$$

i.e., the potential energy depends on time.

If the potential energy is independent of time, then the mechanical energy (T+U) is a constant of the motion; i.e., energy is conserved.

Proof.

$$E = T + U$$

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU(\vec{r})}{dt}$$

$$= \frac{1}{2} m \vec{v} \cdot \frac{d\vec{v}}{dt} + \underbrace{\frac{\partial U}{\partial \vec{r}}}_{\text{This is } \nabla U} \cdot \frac{\partial \vec{r}}{\partial t}$$

$$= m \vec{v} \cdot \vec{a} + \nabla U \cdot \vec{v}$$

$$= \vec{v} \cdot \vec{F} - \vec{F} \cdot \vec{v} = 0$$

dE/dt = 0 so E is constant.

But if U depends on time, then T+U is not a constant of the motion.

$$\frac{dE}{dt} = m \vec{v} \cdot \vec{a} + \frac{\partial U}{\partial t} + \nabla U \cdot \frac{d\vec{r}}{dt}$$

$$= \vec{v} \cdot \vec{F} + \frac{\partial U}{\partial t} - \vec{F} \cdot \vec{v}$$

$$= \frac{\partial U}{\partial t} \quad \leftarrow \boxed{\text{not } 0 \text{ if } U = U(t, \vec{r})}$$

Conservation of energy

"Conservation of energy" is a universal principle of physics. Is there a contradiction here? No, because ….

If U depends on time, then energy must be changing in other parts of the full system.

Conservation of energy

For an isolated system,
the total energy is constant.

(first law of thermodynamics)

But be careful; the total energy must include *all forms of energy* that can contribute to the system.

$E = T + U = \frac{1}{2} m v^2 + U(\boldsymbol{r})$ is the "mechanical energy" of a particle.

E is constant if U does not depend on time.

However, the particle by itself is not an "isolated system", because there must be something else that exerts the force $\boldsymbol{F} = -\nabla U$. Is the other energy changing?

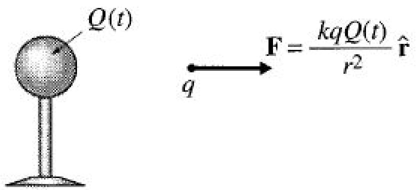


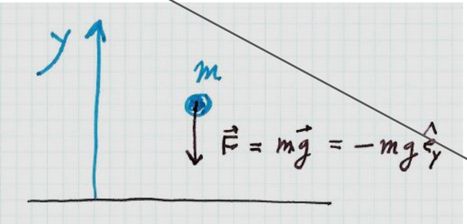
Figure 4.8 The charge $Q(t)$ on the conducting sphere is slowly leaking away, so the force on the small charge q varies with time, even if its position $\mathbf{r}$ is constant.

$$U(r,t) = k\, q\, Q(t)/r$$

Q is changing because electrons are "leaking away" from the sphere. There is energy associated with those electrons.

The mechanical energy of q changes; but the total energy of the system is constant.

Another example: Taylor problem 4.26.



But now suppose $g = g_0\, e^{-\lambda t}$;
i.e., gravity is getting weaker as time passes.

$\boldsymbol{F}$ = mg (downward) = − m g $\boldsymbol{e}_y$

$\boldsymbol{F} = -\nabla\, mgy = -\nabla U$ w/ U = mgy

Suppose a mass m is dropped from rest from initial height y_0.

Calculate $\frac{1}{2}\, m\, y^2 + U(y) = E$.

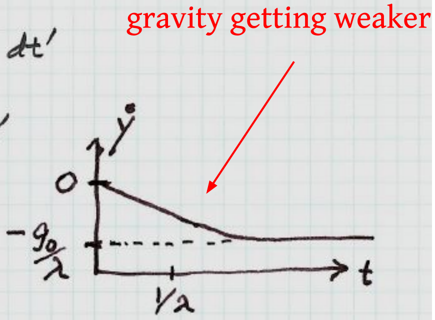
Is the mechanical energy constant?

$$m\ddot{y} = -mg \quad \text{or} \quad \ddot{y} = -g$$

$$\dot{y} = \dot{y}(0) + \int_0^t [-g(t')]\, dt'$$

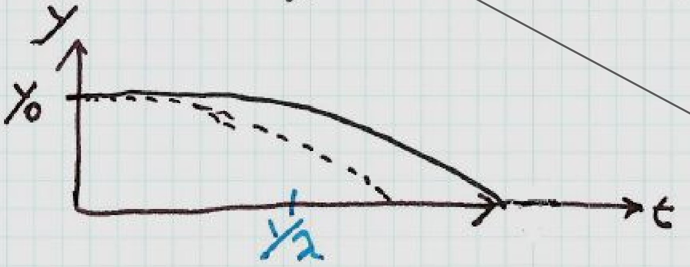
$$= 0 - g_0 \int_0^t e^{-\lambda t'}\, dt'$$

$$= \frac{g_0}{\lambda}\left[e^{-\lambda t} - 1\right]$$



$$y = y_0 + \int_0^t \frac{g_0}{\lambda}\left[e^{-\lambda t'} - 1\right] dt'$$

$$= y_0 - \frac{g_0}{\lambda^2}\left(e^{-\lambda t} - 1\right) - \frac{g_0 t}{\lambda}$$

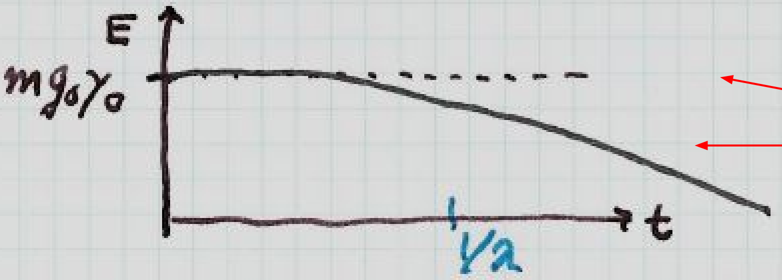


$$E = \frac{1}{2} m \dot{y}^2 + m g y$$

$$= \frac{m}{2}\left(\frac{g_0}{\lambda}\right)^2 \left[e^{-\lambda t} - 1\right]^2 + m g_0 e^{-\lambda t} \left[y_0 - \frac{g_0}{\lambda^2}\left(e^{-\lambda t} - 1\right) - \frac{g_0}{\lambda} t \right]$$

$$= m\left(\frac{g_0}{\lambda}\right)^2 \left\{ \frac{1}{2}\left(e^{-\lambda t} - 1\right)^2 + \frac{\lambda^2 y_0}{g_0} - e^{-\lambda t}\left(e^{-\lambda t} - 1\right) - \lambda t \right\}$$

$$= m g_0 y_0 + \frac{m g_0^2}{\lambda^2} \left\{ -\frac{1}{2}\left(e^{-2\lambda t} - 1\right) - \lambda t \right\}$$



energy would be conserved;
but energy is not conserved.

Symmetries and Conservations Laws

A very general principle in modern theoretical physics …

> For every *symmetry* there is a *conserved quantity*.

One example is:

> *translation invariance in time* implies conservation of *energy*.

symmetry	conservation law
time translation	energy
spatial translations	momentum
rotations	angular momentum
phase transformations	electric charge
QCD gauge transformations	QCD charges

4.6. ENERGY FOR LINEAR MOTION

Linear : $W(x_1 \to x_2) = \int_{x1}^{x2} F_x(x)\, dx$

If F depends only on x then the force is automatically conservative.
Proof is based on Figure 4.9.

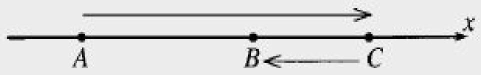


Figure 4.9 The path called *ABCB* goes from *A* past *B* and on to *C*, then back to *B*.

W(ABCB) = W(AB) + W(BC) + W(CB)

= W(AB) ; *thus, path independent*

(*) Linear means one-dimensional.
One-dimensional does not necessarily mean linear; e.g., Section 4.7 on curvilinear motion.

The potential energy function

$$U(x) = -\int_{x^0}^{x} F_x(x')\, dx'$$

The position x_0 is called the "reference point"; it's the position where U = 0.

(The condition $\Delta U = -W$, only defines U up to an additive constant. But if we specify a reference point, i.e., where U = 0, then U is completely defined.)

Example: Hooke's law

A spring acts on a mass constrained to move on the x axis.
⇒ Hooke's law, $F_x(x) = -kx$;
and $U(x) = ½\, kx^2$ because

$$½\, k\, x^2 = -\int_0^x (-k\, x')\, dx'$$

(Note: equilibrium is x = 0 ; that's the ref. point.)

Graph of the potential energy function versus x (*linear motion*)

Interpretation

E = T+U(x) is a constant of the motion.

Because T ≧ 0, U(x) must be ≦ E; *so the particle can only go where U(x) ≦ E.*

Also, where U(x) = E, the velocity must be 0; there x is a *"turning point"*.

Figure 4.10

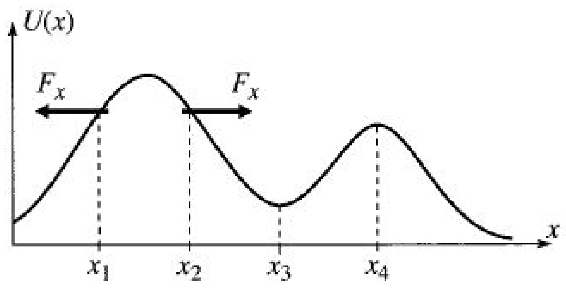


Figure 4.10 The graph of potential energy $U(x)$ against x for any one-dimensional system can be thought of as a picture of a roller coaster track. The force $F_x = -dU/dx$

Figure 4.11

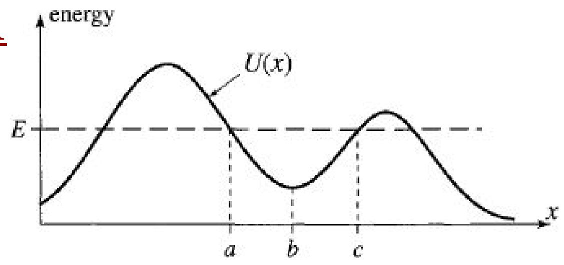


Figure 4.11 If an object starts out near $x = b$ with the energy E shown, it is trapped in the valley or "well"

Figure 4.12

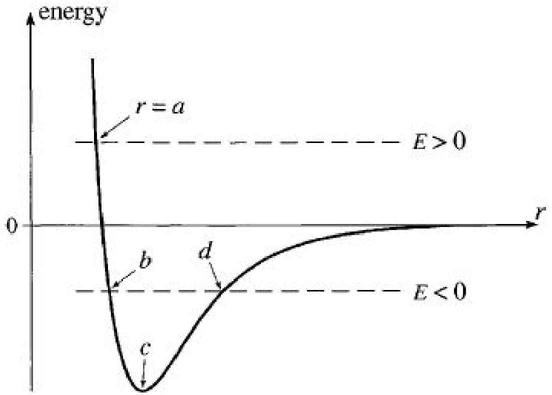


Figure 4.12 The potential energy for a typical diatomic molecule such as HCl, plotted as a function of the distance r between the two atoms. If $E > 0$, the two atoms cannot

Complete solution of the motion

Energy is the first integral of Newton's second law.

Given $m\ddot{x} = F(x)$.

Multiply both sides of the equation by $\dot{x}$ $\Rightarrow$

$$m\dot{x}\ddot{x} = \dot{x}F(x)$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \frac{d}{dt}\left(-U(x)\right)$$

$$\frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} = -F\dot{x}$$

$$\frac{1}{2}m\dot{x}^2 + U(x) = \text{constant} = E$$

energy

$$\dot{x} = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(x)}$$

first integral

Now, $\dot{x} = \frac{dx}{dt}$

Separation of variables $dt = \frac{dx}{\dot{x}}$

Integrate

$$\int_{t_0}^{t} dt' = \int_{x_0}^{x} \frac{dx'}{\pm\sqrt{\frac{2}{m}}\sqrt{E - U(x')}}$$

$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}}$$

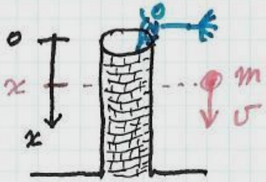
t as a function of x.

Do the integral; then solve for x as a function of t.

second integral

Example 4.8 *free fall*

Drop a stone from a tower at time t = 0. Neglecting air resistance, determine $x(t)$ *from conservation of energy*.



x axis is downward.

$U(x) = -mgx$

$E = 0$

$\frac{1}{2}mv^2 - mgx = 0$ so $\underbrace{v}_{dx/dt} = \sqrt{2gx}$

$$dt' = \frac{dx'}{\sqrt{2gx'}}$$

$$t = \frac{1}{\sqrt{2g}} \int_0^x x'^{-1/2}\,dx' = \frac{1}{\sqrt{2g}}\, 2x^{1/2} = \sqrt{\frac{2x}{g}}$$

$$x = \frac{1}{2}gt^2$$ (of course)

Test yourself

An object with mass = m moves on the x axis with potential energy $U(x) = \frac{1}{2} k x^2$. The initial values are $x_0 = -1$ m and $v_0 = 2$ m/s.

Calculate the maximum x that it will reach.

Homework Assignment #7
due in class Wednesday, October 18

[31] Problem 4.3 **
[32] Problem 4.8 **
[33] Problem 4.9 **
[34] Problem 4.10 *
[35] Problem 4.18 **
[36] Problem 4.23 **

Use the cover page.

This is a pretty long assignment, so do it now.