4.8. Central forces

The most interesting problems in classical mechanics are about central forces.

Definition of a central force:

(i) the *direction* of the force ***F(r)*** is parallel or antiparallel to ***r*** ***;***

in other words, for any position *of the object on which F is acting*, the direction of ***F*** points toward or away from the origin.

{attraction to O or repulsion from O}

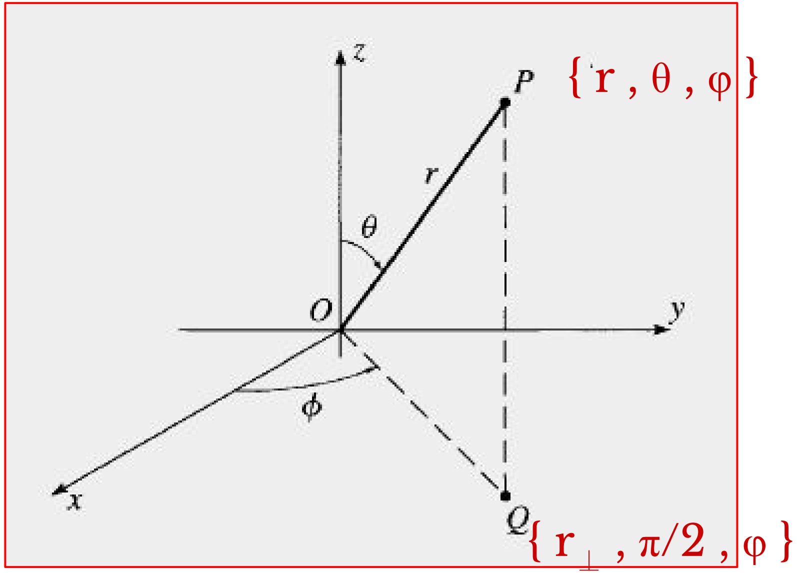
(ii) the *magnitude* of the force **| *F(r)* |** depends only of the distance |***r***| from the origin. ***{spherically symmetric}***

$$\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \hat{\mathbf{r}}\, f\,(r)$$

The simplest way to analyze the motion of the object is to use *spherical polar coordinates*.

Spherical Polar Coordinates {r,θ,φ}

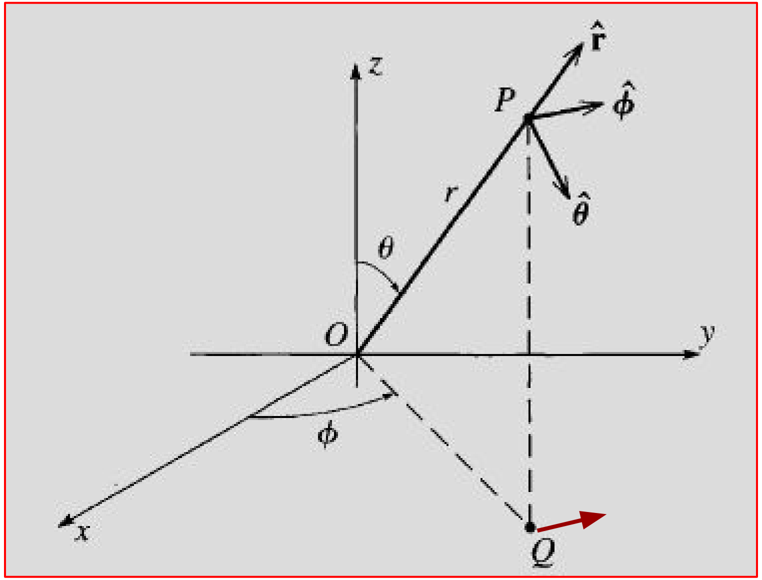
Figure 4.16



Spherical Polar Coordinates

The *direction vectors* for spherical polar coordinates are $\hat{\mathbf{r}}$ $\hat{\boldsymbol{\theta}}$ $\hat{\boldsymbol{\varphi}}$

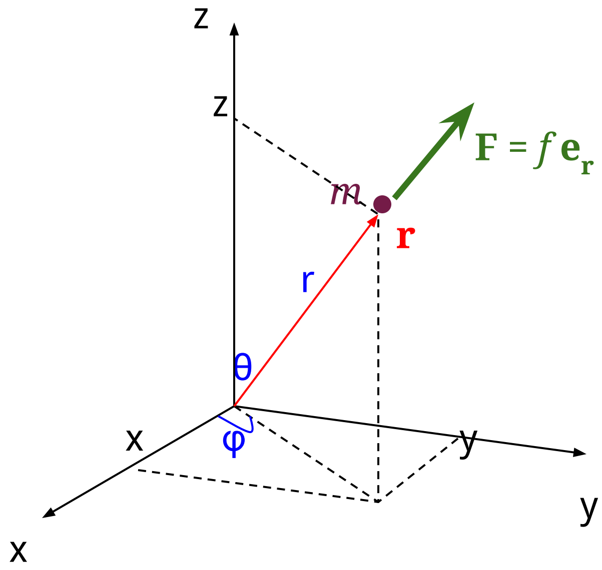
Figure 4.17



A central and spherically symmetric force, using spherical polar coordinates

$$\vec{F}(\vec{r}) = f(r)\,\hat{e}_r$$

$$m\ddot{\vec{r}} = f(r)\,\hat{e}_r \qquad \text{where } \vec{r} = r\,\hat{e}_r$$



"The origin is the center of force."

Conservative central forces

$$\mathbf{F(r)} = -\nabla U(r)$$

Or, handwritten,

$$\vec{F}(\vec{r}) = -\nabla U(r)$$

In words, the potential energy function U (which is a scalar) depends only on the distance from the center of force, *r*.

U(x,y,z) = U(*r*) where $r = \sqrt{x^2+y^2+z^2}$

U(r) is called a *spherically symmetric potential.*

You should know the gradient operator in spherical polar coordinates,

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

(*study the formulas inside the back cover of the book*)

So, for a spherically symmetric potential, U(r)

$$\nabla U = \hat{r}\frac{dU}{dr}$$

$F_r(r) = -\,dU/dr$

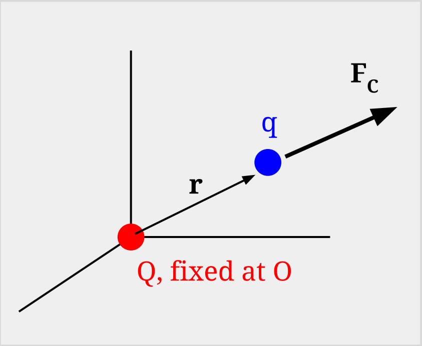
The Coulomb force and the corresponding potential energy

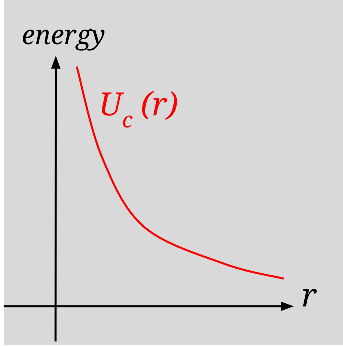
$$\vec{F}_c(\vec{r}) = \frac{kQq}{r^2}\,\hat{e}_r$$

$$k = 8.99 \times 10^9\ \mathrm{Nm^2/C^2}$$
$$= 8.99 \times 10^9\ \mathrm{Vm/C}$$

$$\vec{F}_c = -\nabla U_c = -\hat{e}_r \frac{dU_c}{dr}$$

$$\therefore\ U_c(r) = \frac{kQq}{r}$$





Example. Hydrogen atom.

⊕ ----- r -----> ⊖

p, Q   e, q

$Q = e$

$q = -e$

$$U_c = -\,8.99 \times 10^9\ \frac{\mathrm{Vm}}{\mathrm{C}}\ e\ 1.6 \times 10^{-19}\,\mathrm{C}\,/\,r$$

$$U_c = \frac{-1.438}{r}\ \mathrm{eV \cdot nm}$$

NEWTON'S THEORY OF UNIVERSAL GRAVITATION

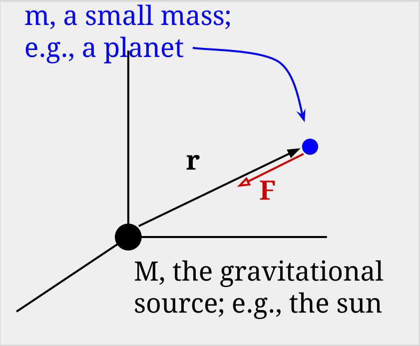
Imagine a large mass M (like the sun; or the Earth) and a smaller mass m (like a planet; or a satellite).
The force on mass m is

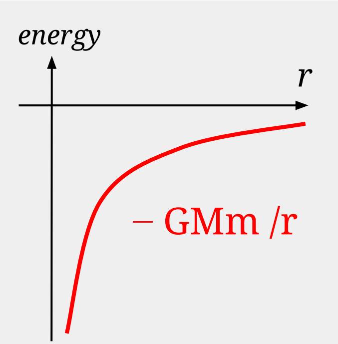
$$\mathbf{F} = -\frac{GMm}{r^2}\,\mathbf{e_r}$$

More complete …

- *Both masses are attracted toward each other —— Newton's third law.*
- *Both masses revolve around the CoM, and the CoM is fixed.*

Chapter 8





The potential energy function is

$$U(r) = -\,GMm/r$$

where $G = 6.67 \times 10^{-11}\ m^3\ s^{-2}\ kg^{-1}$.

We have a spherically symmetric potential *assuming the bodies are themselves spherically symmetric*.

*The Earth is slightly oblate, so the potential energy of the Earth-Moon system is not **truly** central.*

Furthermore…

There are tides.

- Synchronicity of the Moon

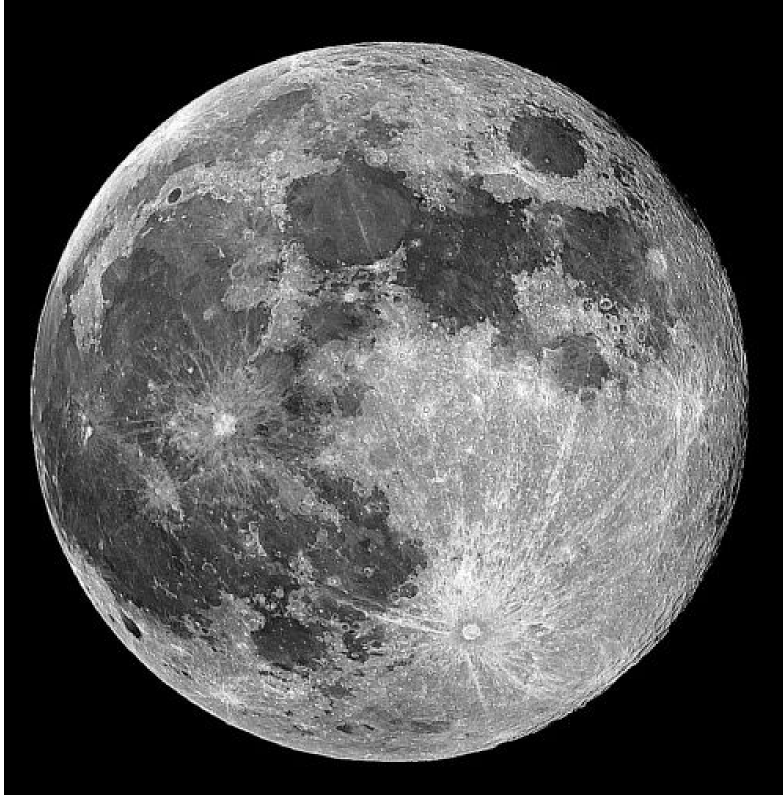
  $\tau_{revolutions} = \tau_{rotations\ Moon}$;

  reason … because of ancient tides in the Moon; "tidal evolution".

- Today tides in the Earth are dissipating energy; ∴ the orbit radius is increasing and the period of rotation of the Earth is increasing.

- Angular momentum is conserved,

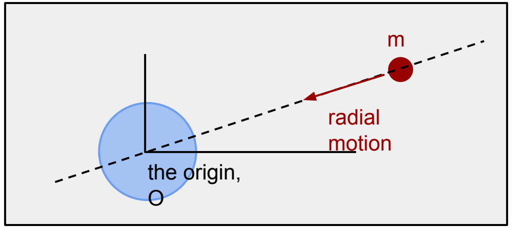
  $\dot{L}_{rotE} + \dot{L}_{revE} = 0.$



Tidal evolution of the Earth moon system today …

$\dot{r} = 38$ mm /yr and $\dot{\tau}_{rotE} = 23\ \mu s$ /yr

Radial motion in a spherically symmetric potential



This is an example of "1-dimensional motion" in three dimensions.

If the angular momentum is 0 then the mass m just moves on a radial line through the origin.

Example

An asteroid falls radially toward the Earth, with zero angular momentum relative the Earth.

Assume these parameters:

- When the asteroid is at distance r_0 from the Earth (r = distance from the center of the Earth) the velocity is 0.
- Pick a number : let's try

r_0 = earth-moon distance = 384,000 km.

Never mind how this unusual initial condition might be created; just take it as given.

- The asteroid mass; suppose R_a = 1 km ; then the mass is

$$m = 4/3\ \pi\ R_a^{\ 3}\ .\ (5\times10^3\ \text{kg/m}^3) = 2\ \times10^{13}\ \text{kg}$$

First question …

Calculate the *kinetic energy* of the asteroid when it hits the Earth.

Solution

For this we only need an algebraic calculation using conservation of energy …

$$E = \tfrac{1}{2}\, m\, v^2 - GMm/r$$

$$\therefore\ T_{hit} - GMm/R_\oplus = -\,GMm/r_0$$

$$T_{hit} = GMm\,(1/R_\oplus - 1/r_0)$$

$GM_\oplus/R_\oplus^{\,2} = g$ and $1/r_0 \approx 0$, so

$$T_{hit} = m\, g\, R_\oplus = 2E13 \times 9.8 \times 6.4E6 \text{ J}$$

$$T_{hit} = 1.2 \times 10^{21} \text{ J} = 1.2 \times 10^{6} \text{ petajoules}$$

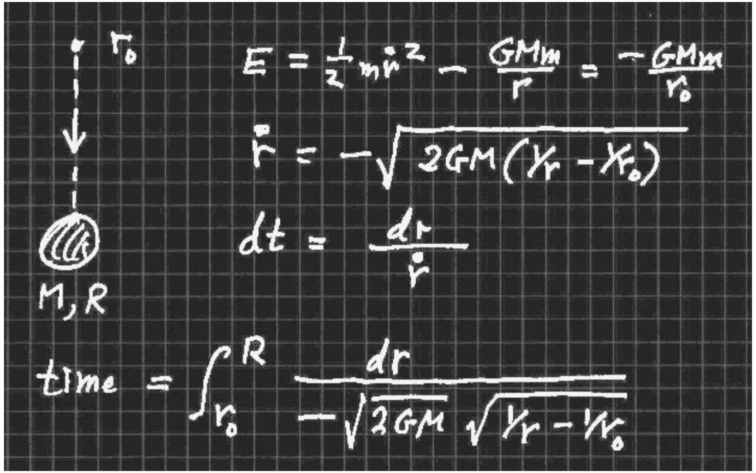
Compare largest ever H bomb = "Tsar Bomba" = 50 megaton TNT = 210 PJ

Second question …

Calculate the *time it will take* for the asteroid to fall to the surface of the Earth, from the initial distance r_0 .

Solution

This is a ***time calculation***, so we need to solve a differential equation.



$$\text{time} = \int_{r_0}^{R} \frac{dr}{-\sqrt{2GM}\,\sqrt{1/r - 1/r_0}}$$

$$= \frac{1}{\sqrt{2GM}} \int_{R}^{r_0} \sqrt{\frac{r r_0}{r_0 - r}}\, dr$$

$$r = r_0 u$$

$$= \sqrt{\frac{r_0^3}{2GM}} \underbrace{\int_{R/r_0}^{1} \sqrt{\frac{u}{1-u}}\, du}_{\approx \frac{\pi}{2} \text{ because } \frac{R}{r_0} \approx 0}$$

$$\text{time} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}} = 4.17 \times 10^5 \text{ sec.}$$

$$= 4.83 \text{ days}$$

$$\int \sqrt{\frac{u}{1-u}}\, du = \arcsin(\sqrt{u}) - \sqrt{u(1-u)} + C$$

Interesting comparisons:
Moon to Earth travel time (from TEI SPS ignition to splashdown):
{ Trans‧Earth‧Injection ; Service‧Propulsion‧System }
Apollo 11: 2 days 11 hours 55 minutes
Apollo 12: 3 days 9 minutes
Apollo 14: 2 days 19 hours 26 minutes
Apollo 15: 2 days 23 hours 23 minutes
Apollo 16: 2 days 17 hours 30 minutes
Apollo 17: 2 days 19 hours 49 minutes

Homework Assignment #8
due in class Wednesday, October 25
[37] Problem 4.26 *
[38] Problem 4.28 ** and Problem 4.29 ** [Computer]
[39] Problem 4.33 ** [Computer]
[40] Problem 4.34 **
[41] Problem 4.37 *** [Computer]
[42] Problem 4.38 *** [Computer]
Use the cover page.
This is a long assignment, so start working on it now.