5.3. Two dimensional oscillators

The definition of an "isotropic" oscillator in 2 or 3 dimensions is

\[ F = -k r \]

\[ U = \frac{1}{2} k r^2 = \frac{1}{2} k (x^2 + y^2 + z^2) \]

**in 3 dimensions**

Figure 5.7 shows a 2d example; the particle (mass m) attached to the 4 springs moves in the xy plane.
Comments about Figure 5.7.
The particle (mass = m) attached to the springs moves in the xy plane.

What is the potential energy when the particle is displaced to \( \{ x, y \} \)?

Assume that the equilibrium length of each spring is \( a \), and the spring constant is \( k/2 \). Also, the size of the square is \( 2a \times 2a \).

\[
\mathcal{U} = \frac{1}{2} k \left( l_1 - a \right)^2 + \frac{1}{4} k \left( l_2 - a \right)^2 + \frac{1}{4} k (l_3 - a)^2 + \frac{1}{4} (l_4 - a)^2
\]

\[
l_1 = \sqrt{(a-x)^2 + y^2} \quad \approx a - x + \frac{y^2}{2a}
\]

\[
l_2 = \sqrt{x^2 + (a-y)^2} \quad \frac{a}{2} \frac{c}{l_2} \frac{b}{l_4}
\]

\[
\mathcal{U} \approx \frac{1}{2} k (x^2 + y^2) \quad 4 k^2 y^2 < a^2
\]

\[
= \frac{1}{2} ker^2 \quad \text{if} \quad \mathcal{E} = -k r^2
\]

\[^{\text{It is perhaps worth pointing out that one does not get a force of the form (5.17) by simply attaching a mass to a spring whose other end is anchored to the origin.}}\]
Three examples of *isotropic* oscillations in 2d:

\[ U = \frac{1}{2} k x^2 + \frac{1}{2} k y^2 \]

\[ x(t) = A \cos(\omega t) \]

\[ y(t) = B \cos(\omega t - \delta) \]

- (a) \( \delta = 0 \)
- (b) \( \delta = \pi/2 \)
- (c) \( \delta = \pi/4 \)

\[ i.e., k_x = k_y \]
Figure 5.9.
Two examples of anisotropic oscillations

\[ i.e., k_x \neq k_y \]

\[ U = \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2 \]

\[ x(t) = A \cos(\omega_x t) \]

\[ y(t) = B \cos(\omega_y t - \delta) \]

(a) 2x1 Lissajous fig.  
(b) quasi periodic

(a) \( \omega_x = 2 \omega_y \)  
(b) \( \omega_x = \sqrt{2} \omega_y \)
5.4. Damped oscillations

Sometimes in everyday life, oscillations may create problems.

For example, that's why a car has shock absorbers — to damp out the oscillations when the wheels hit a bump in the road, or a pothole.

Go back to 1-dimensional oscillations, but now add damping.

**Generic picture**

The equation of motion is

\[ m \ddot{x} = -b \dot{x} - k x \]

Note the assumption of "linear damping"; i.e., \( F_{\text{damping}} = -b \dot{x} \);

or, we can write it this way,

\[ m \dddot{x} + b \dot{x} + k x = 0. \]

It is useful to "rescale the parameters" to write the equation in a standard form;

\[ x + 2\beta x + \omega_0^2 x = 0 \]

where

\[ 2\beta = \frac{b}{m} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}. \]
Solution. This is an example of a "homogeneous linear differential equation with constant coefficients". There is a standard method to solve this kind of diff. eq. (MTH 234)

First, try \( x(t) = e^{pt} \).
\[ \dot{x} = p e^{pt} \quad \text{and} \quad x'' = p^2 e^{pt}, \] so
\[ p^2 + 2\beta p + \omega_0^2 = 0. \]
\[ p_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \]
Overdamped oscillator: \( \beta > \omega_0 \)

This is the case of strong damping. In this case \( p_+ \) and \( p_- \) are real.

\[
x(t) = c_+ e^{p_+ t} + c_- e^{p_- t} = e^{-\beta t} \left[ c_+ e^{\sqrt{\beta^2 - \omega_0^2} t} + c_- e^{-\sqrt{\beta^2 - \omega_0^2} t} \right]
\]

Underdamped oscillator: \( \beta < \omega_0 \)

This is the case of weak damping. In this case \( p_1 \) and \( p_2 \) are complex numbers.

Recall \( e^{\pm i \theta} = \cos \theta \pm i \sin \theta \) (Euler)

\[
x(t) = e^{-\beta t} \left[ A \cos \omega_1 t + B \sin \omega_1 t \right]
\]

where \( \omega_1 = \sqrt{\omega_0^2 - \beta^2} \).

\[
x(0) = A \quad \text{and} \quad \dot{x}(0) = -\beta A + \omega_1 B
\]
The critically damped oscillator

$\beta = \omega_0$

In this case $p_+$ and $p_-$ are equal, $p_+ = p_- = \omega_0$; so $\exp(pt)$ is only one solution. To get the general solution we need another solution.

Exercise: Show that $x(t) = t \exp(pt)$ is also a solution for the critically damped oscillator ($\beta = \omega_0$).

Example. Consider these initial conditions: $x(0) = 1$ and $v(0) = 0$. 
Fig. 5.12 corresponds to these initial conditions: \( x(0) = 0 \) and \( v(0) > 0 \); i.e., 1- the mass is kicked in the +x direction, 2- it reaches a maximum displacement, and 3- it returns to equilibrium monotonically.
Critical damping \((\beta = \omega_0)\)
This special case has the most rapid return to equilibrium ...

The "decay parameter" \(p\) versus \(\beta\). The decay parameter is largest---so the motion dies out most quickly---for critical damping \(\beta = \omega_0\).

<table>
<thead>
<tr>
<th>damping</th>
<th>(\beta)</th>
<th>decay parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>(\beta = 0)</td>
<td>0</td>
</tr>
<tr>
<td>under</td>
<td>(\beta &lt; \omega_0)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>critical</td>
<td>(\beta = \omega_0)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>over</td>
<td>(\beta &gt; \omega_0)</td>
<td>(\beta - \sqrt{\beta^2 - \omega_0^2})</td>
</tr>
</tbody>
</table>
A mass $m$ moves in the xy-plane, attached to a spring as shown. According to a footnote in Taylor, the force on $m$ is not $-kr$. OK, then, what is the force?

Homework Assignment #9
due in class Wednesday November 1
[41] Problem 4.41 and Problem 4.43
[42] SEE THE COVER SHEET
[43] Problem 5.3 *
[44] Problem 5.5 *
[45] Problem 5.9 *
[46] Problem 5.12 **
[47] Problem 5.18 ***

Use the cover sheet.