Section 5.7

Fourier Series

Motivation

Consider the driven damped oscillator, but for an *arbitrary periodic driving force*.

Recall, $D\,x(t) = f(t)$

where $D\,x = \ddot{x} + 2\,\beta\,\dot{x} + \omega_0^{\,2}\,x$

and $f = F/m$.

"Periodic" means $f(t + \tau) = f(t)$.

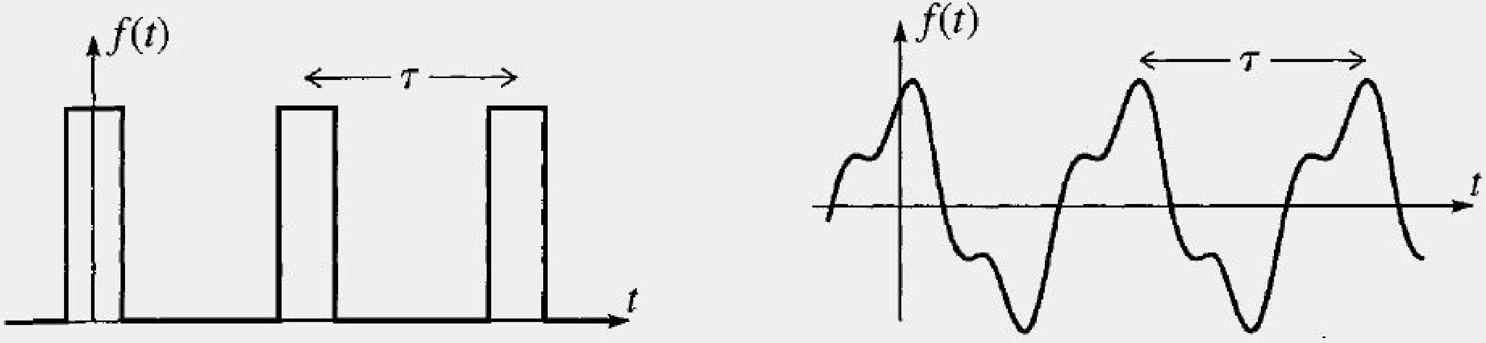
Now, "periodic" does not necessarily mean sinusoidal.
("Sinusoidal" and "harmonic" are the same thing.)

A harmonic function is periodic ;
e.g., $f(t) = f_0 \cos \omega t$ is periodic, with period $\tau = 2\pi / \omega$;
proof :

$$\begin{aligned} f(t + \tau) &= f_0 \cos[\omega(t + \tau)] \\ &= f_0 \cos[\omega t + 2\pi] \\ &= f_0 \cos \omega t \qquad = f(t). \end{aligned}$$

But a periodic function is not necessarily harmonic.

Figure 5.20 . Two examples of periodic functions



(a) (b)

Figure 5.20 Two examples of periodic functions with period τ. **(a)** A rectangular pulse, which could represent a hammer hitting a nail with a constant force at intervals of τ, or a digital signal in a telephone line. **(b)** A smooth periodic signal, which could be the pressure variation of a musical instrument.

Motivation

To solve :

$$D\,x(t) = f(t)$$

where $D\,x = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$

and $f(t) = F(t)/m$ is periodic

$$f(t+\tau) = f(t).$$

- We know the solution if $f(t)$ is harmonic (***from the previous lecture***).

- Now consider $f = f_1 + f_2$; then $x = x_1 + x_2$, because the equation is linear.

- So, if $f(t)$ is a superposition of harmonic functions, then x(t) is the superposition of corresponding solutions, which we already know.

- <u>Fourier's theorem</u>

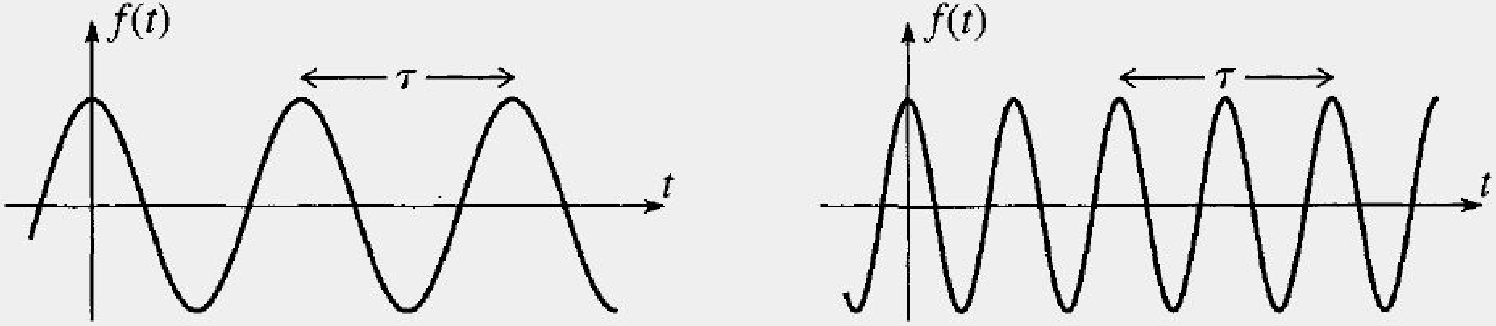
If $f(t)$ is a periodic function, then it can be written as a superposition of harmonic functions,

$$f(t) = \sum_{n=0}^{\infty} [\, a_n \cos(n\omega t) + b_n \sin(n\omega t) \,]$$

where $\omega = 2\pi/\tau$.

Easy exercise:: Prove that f (t) is periodic.

Figure 5.21



$\cos(2\pi t/\tau)$ $\cos(4\pi t/\tau)$

Figure 5.21 Any function of the form $\cos(2n\pi t/\tau)$ (or the corresponding sine) is periodic with period τ if n is an integer. Notice that $\cos(4\pi t/\tau)$ also has the smaller period $\tau/2$, but this doesn't change the fact that it has period τ as well.

▌ Fourier's theorem

If $f(t)$ is a periodic function, then it can be written as a superposition of harmonic functions.

$$f(t) = \sum_{n=0}^{\infty} [\, a_n \cos(n\omega t) + b_n \sin(n\omega t) \,]$$

where $\omega = 2\pi / \tau$.

***Proof:* Take a math course.**

Now,
given $f(t)$, what are the coefficients a_n and b_n ?

Simplifications:

※ If f(t) is an even function of t,

i.e., $f(-t) = f(t)$, then

$b_1 = 0 = b_2 = b_3 = \ldots\ b_n = \ldots = 0.$

※ If f(t) is an odd function of t,

i.e., $f(-t) = -f(t)$, then

$a_0 = 0 = a_1 = a_2 = \ldots\ a_n = \ldots = 0.$

※ Today, we'll assume $f(t)$ is even;

$\Rightarrow$ a superposition of ***cosines*** .

Example 5.4 :

periodic rectangular pulses

Figure 5.22

Important: understand that this *f(t)* is an even function of t.

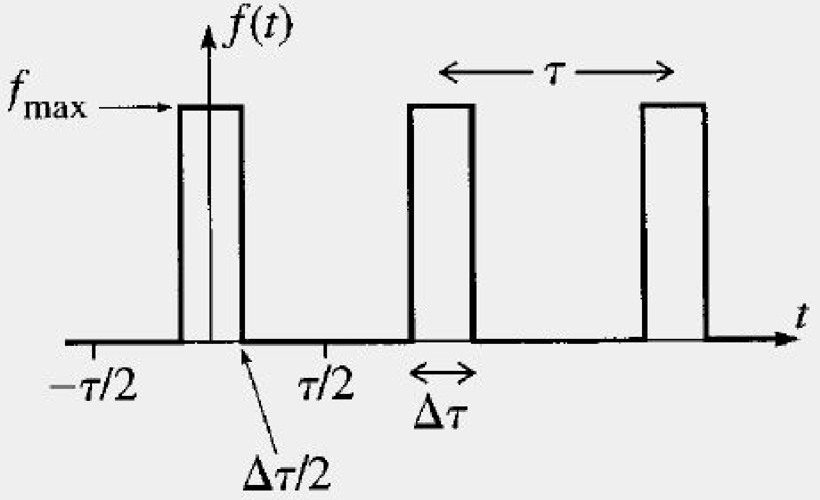


Figure 5.22 A periodic rectangular pulse. The period is τ, the duration of the pulse is $\Delta\tau$, and the pulse height is f_{max}.

Parameters:

τ

$\Delta\tau$

f_{max}

The *Fourier coefficients* for an even periodic function …

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) \quad \text{where } \omega = \frac{2\pi}{\tau}$$

even function of t : $f(-t) = f(t)$

Now consider

$$\bigstar = \int_{-\tau/2}^{\tau/2} f(t) \cos(n\omega t)\, dt$$

$$= \sum_{n'=0}^{\infty} a_{n'} \underbrace{\int_{-\tau/2}^{\tau/2} \cos(n'\omega t) \cos(n\omega t)\, dt}$$

$$= \int_{-\tau/2}^{\tau/2} \left\{ \cos(n'+n)\omega t + \cos(n'-n)\omega t \right\} \frac{dt}{2}$$

$$= \frac{1}{(n'+n)\omega} \sin\left[(n'+n)\omega t\right] + \frac{1}{(n'-n)\omega} \sin\left[(n'-n)\omega t\right] \Bigg|_{t=-\tau/2}^{t=\tau/2} \times \frac{1}{2}$$

$$\omega \tau/2 = \pi$$

$$\sin(-x) = -\sin(x)$$

$$= \frac{\sin[(n'+n)\pi]}{(n'+n)\pi} + \frac{\sin[(n'-n)\pi]}{(n'-n)\pi} = 0 \text{ if } n' \neq n$$

$$\bigstar = a_n \int_{-\tau/2}^{\tau/2} \cos^2(n\omega t)\, dt = \begin{cases} a_0 \tau & \text{case } n=0 \\ a_n \dfrac{\tau}{2} & \text{case } n \neq 0 \end{cases} = \bigstar$$

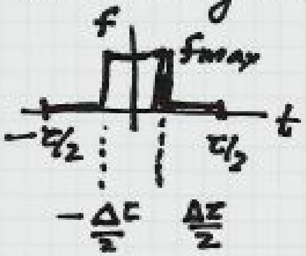
Thus

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\, dt$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos(n\omega t)\, dt \quad \text{for } n > 0.$$

for any <u>even</u> periodic function $f(t)$.

The periodic rectangular pulse



$$\boxed{\omega = \frac{2\pi}{\tau} \qquad \frac{\omega}{2} = \frac{\pi}{\tau}}$$

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt = f_{max}\frac{\Delta\tau}{\tau}$$

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\,dt$$

$$= \frac{2}{\tau} f_{max}\frac{1}{n\omega}\sin(n\omega t)\Big]_{t=-\Delta\tau/2}^{t=\Delta\tau/2}$$

$$= \frac{2}{\tau} f_{max}\frac{2}{n\omega}\sin\left(n\omega\frac{\Delta\tau}{2}\right)$$

$$a_n = f_{max}\frac{2}{n\pi}\sin\left(\frac{n\pi\Delta\tau}{\tau}\right)$$

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t)$$

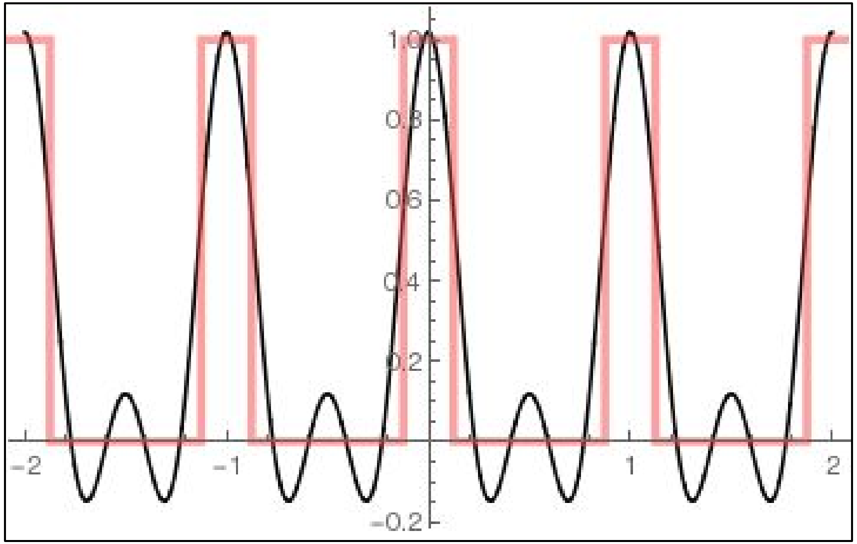
The periodic rectangular pulse.

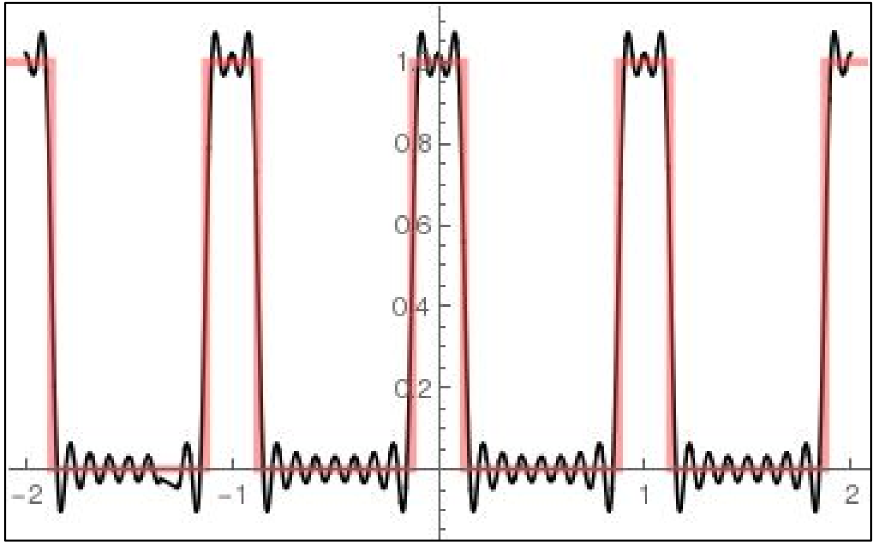
Let's look at it for Δτ = 0.25 τ .

$$\Delta\tau = 0{,}25\,\tau \Rightarrow \begin{cases} a_0 = \dfrac{f_{max}}{4} \\ a_n = f_{max}\dfrac{2\sin\left(\frac{n\pi}{4}\right)}{n\pi} \quad (n>0) \end{cases}$$

n	a_n/f_{max}	
0	$1/4$	$= 0{,}250$
1	$\sqrt{2}/\pi$	$= 0.450$
2	$1/\pi$	$= 0.318$
3	$\sqrt{2}/3\pi$	$= 0.150$
4	0	$= 0.00$
etc		

Fourier series truncated at n = 2 || truncated at n = 10





Comments:

(1) as $N \longrightarrow \infty$ the Fourier series approaches the function f(t);

(2) at a discontinuity, the *truncated* Fourier series can't reproduce the discontinuity.

Figure 5.23 : The Fourier series for the periodic rectangular pulse, truncated to (a) 3 terms , and (b) 11 terms. As N $\rightarrow \infty$, the series approaches f.

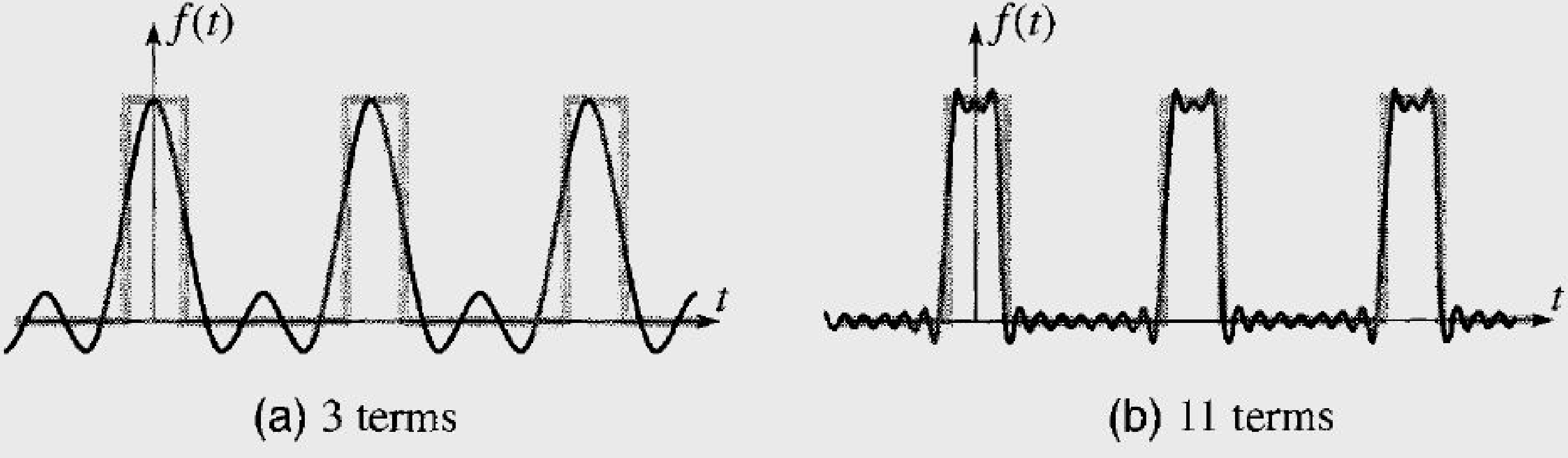


Figure 5.23 **(a)** The sum of the first three terms of the Fourier series for the rectangular pulse of Figure 5.22. **(b)** The sum of the first 11 terms.

Preview of Section 5.8.

Fourier Series Solution
for the Driven Oscillator

To solve, $D\,x = f$.

We'll just obtain the *steady-state solution* ; i.e., the particular solution that x(t) approaches as $t \to \infty$.

We'll have $f(t) = \sum_n a_n \cos(n\omega t)$.

(assuming f(t) is even in t)

By the superposition principle,

$$x(t) = \sum_n a_n A_n \cos[\, n\omega(t - \delta_n)\,] .$$

A_n and δ_n are derived in Section 5.6 (and in Friday's lecture).

Homework Assignment #10
due in class Wednesday November 8

[47] Problem 4.53
[48] Problem 5.25 **
[49] Problem 5.30 **
[50] Problem 5.37 **
[51] Problem 5.44 **
[52] Problem 5.52 *** [Computer]

Exam 2 will be Friday November 3;

- conservation of energy;
- Section 4.6; "complete solution";
- central forces;
- damped oscillations;