Section 5.8

Fourier series solution

for the driven oscillator

Section 5.9

RMS displacement

Read Sections 5.8 and 5.9.

Fourier series solution for the driven oscillator

/1/ To solve: $Dx = f$ (1)

where

$$D = d^2/dt^2 + 2\,\beta\, d/dt + \omega_0^{\,2}$$

and $f(t)$ is a periodic driving force with angular frequency $\omega = 2\pi/\tau$.

(β = damping constant; ω_0 = natural frequency)

We'll just determine the *steady-state solution* ; i.e., the particular solution that x(t) approaches as $t \to \infty$.

/2/ Method:

By Fourier's theorem we can write

$$f(t) = \underbrace{\sum_{n=0}^{\infty} a_n \cos(n\omega t)}_{\text{(even in t)}} + \underbrace{\sum_{n=0}^{\infty} b_n \sin(n\omega t)}_{\text{(odd in t)}}$$

To make it simple, assume $f(t)$ is even; then $b_n = 0$ for all n.

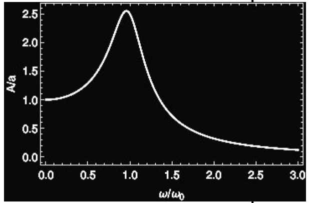
/3/ Recall Section 5.6

Consider the *harmonic* driving force

$$f = a_n \cos(\omega_n t) \qquad [\, \omega_n \equiv n\omega \,]$$

The steady-state solution is already known from Section 5.6: recall,

$$x_n(t) = A_n \cos(\, n\omega\, t - \delta_n \,)$$

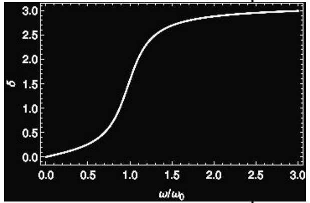


where

$$A_n = \frac{a_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (2\,\beta\, n\omega)^2}}$$

and

$$\delta_n = \arctan\left(\frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}\right)$$



/4/ Superposition

Equation (1) is linear, so… *if*

$$f = \Sigma f_n(t) = \Sigma\, a_n \cos(\, n\omega\, t\,)$$

then

$$x = \Sigma x_n(t) = \Sigma\, A_n \cos(\, n\omega\, t - \delta_n \,)$$

"superposition principle"

"the stationary solution"

Putting it into words:

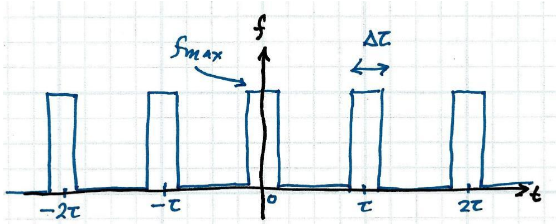
Given the Fourier series for *f* (t) , we obtain the Fourier series for x(t) by superposition, because the equation is linear.

So now we have here the asymptotic behavior of the oscillator; valid as $t \to \infty$; independent of the *transients* , which are damped out by the effect of β.

Example 5.5.

An oscillator driven by periodic rectangular pulses

The forcing function



period = τ

impulse duration = $\Delta\tau$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$$

$$a_0 = \frac{f_{max}\,\Delta\tau}{\tau} \quad \text{and} \quad a_n = \frac{2 f_{max}}{n\pi}\sin\left(\frac{n\pi\,\Delta\tau}{\tau}\right) \quad (n \geq 1)$$

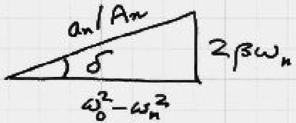
The steady state solution

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

$$A_n = \frac{a_n}{\left[(\omega_0^2 - n^2\omega^2)^2 + 4\beta^2 n^2\omega^2\right]^{1/2}}$$

$$\tan\delta_n = \frac{2\beta n\omega}{\omega_0^2 - n^2\omega^2}$$

THE RESONANCE TRIANGLE ($\omega_n = n\omega$)



Now we need a computer.

Example 5.5: an oscillator driven by a rectangular pulse

Figure 5.24

In Fig. 5.24,
$\tau = \tau_0$;
i.e., the period of the driving force is equal to the natural period,
also, $\Delta\tau = 0.25\,\tau$.

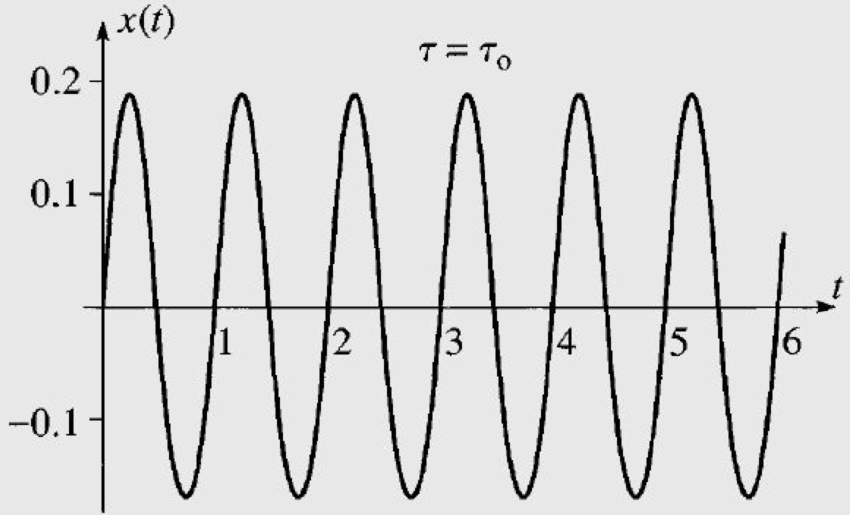


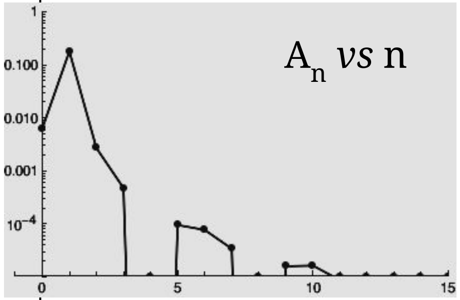
Figure 5.24 The motion of a linear oscillator, driven by periodic rectangular pulses, with the drive period τ equal to the natural period τ_o of the oscillator (and hence $\omega = \omega_o$). The horizontal axis shows time in units of the natural period τ_o. As expected the motion is almost perfectly sinusoidal, with period equal to the natural period.

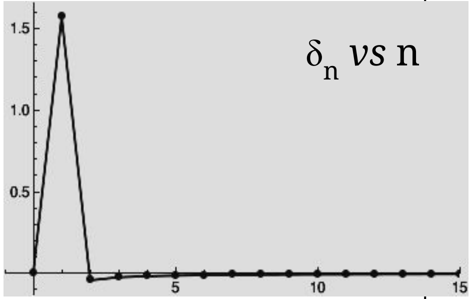
But there is a phase shift of 90 degrees.

Mathematica Calculations …

… to verify Figure 5.24.

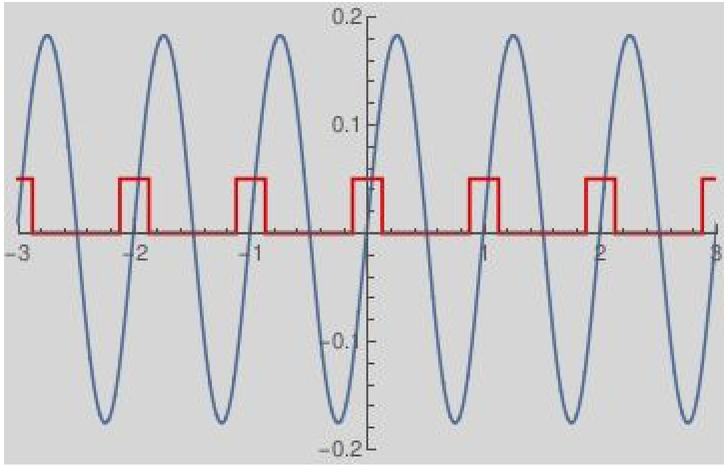
(a) As a first case, set $\omega = \omega_0$.
Then plot the amplitude A_n and phase shift δ_n versus n :





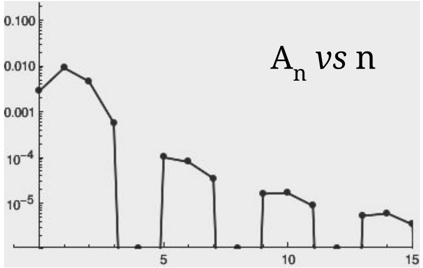
This explains why x(t) closely approximates a harmonic oscillation with frequency ω and phase shift $\pi/2$: *because the Fourier contribution of n = 1 is in resonance,*

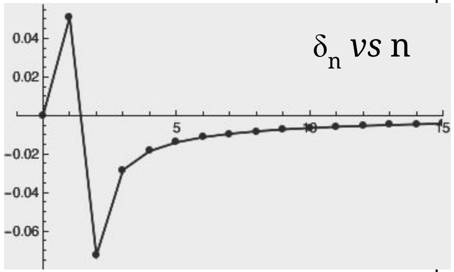
$$\omega_1 = 1\,\omega = \omega_0 .$$

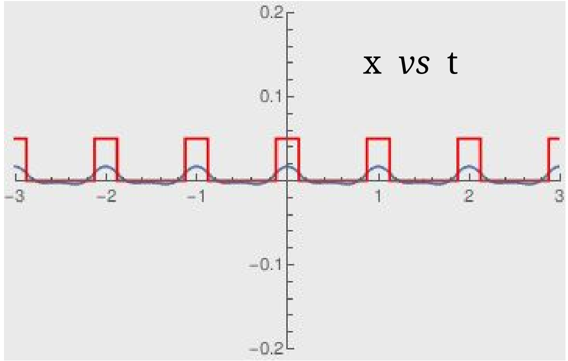


Now consider three other cases.

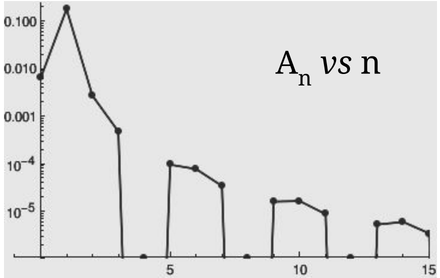
(b) Case $\tau = 1.5\,\tau_0$; i.e., $\omega = 0.667\,\omega_0$; *no Fourier component is in resonance* ; $n\omega = \omega_0$ would mean n = 1.5, but that is not an integer.

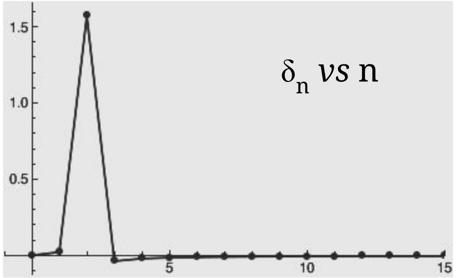


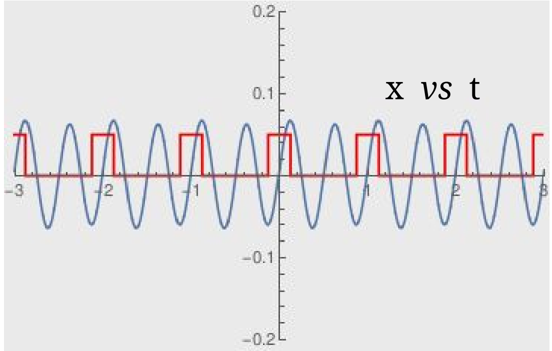




(c) Case $\tau = 2\,\tau_0$; i.e., $\omega = 0.5\,\omega_0$; *the Fourier component with n=2 is in resonance* ; $2\omega = \omega_0$.



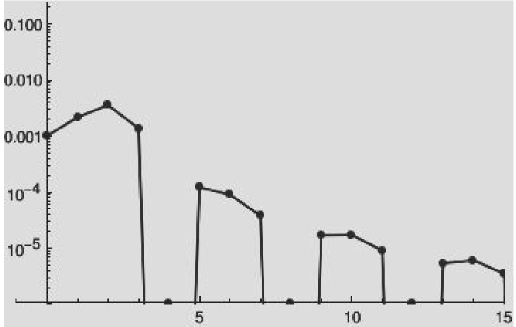


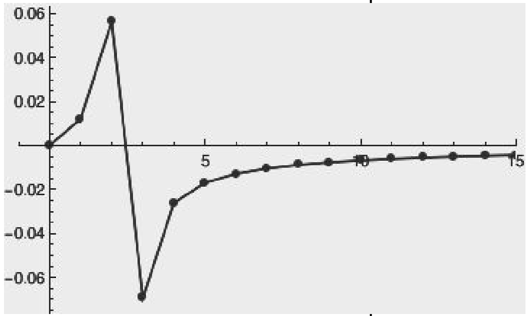


(d) Case $\tau = 2.5\,\tau_0$; i.e., $\omega = 0.4\,\omega_0$;

no Fourier component is in resonance ;

$n\omega = \omega_0$ would mean n = 2.5, but that is not an integer.





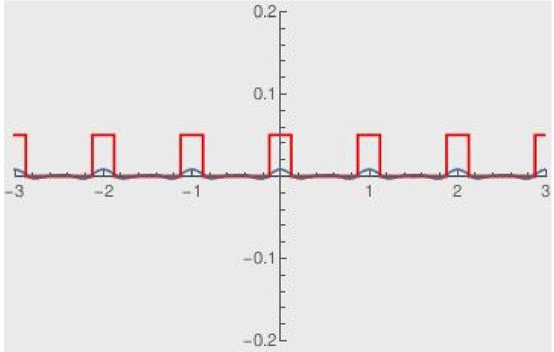


Figure 5.25

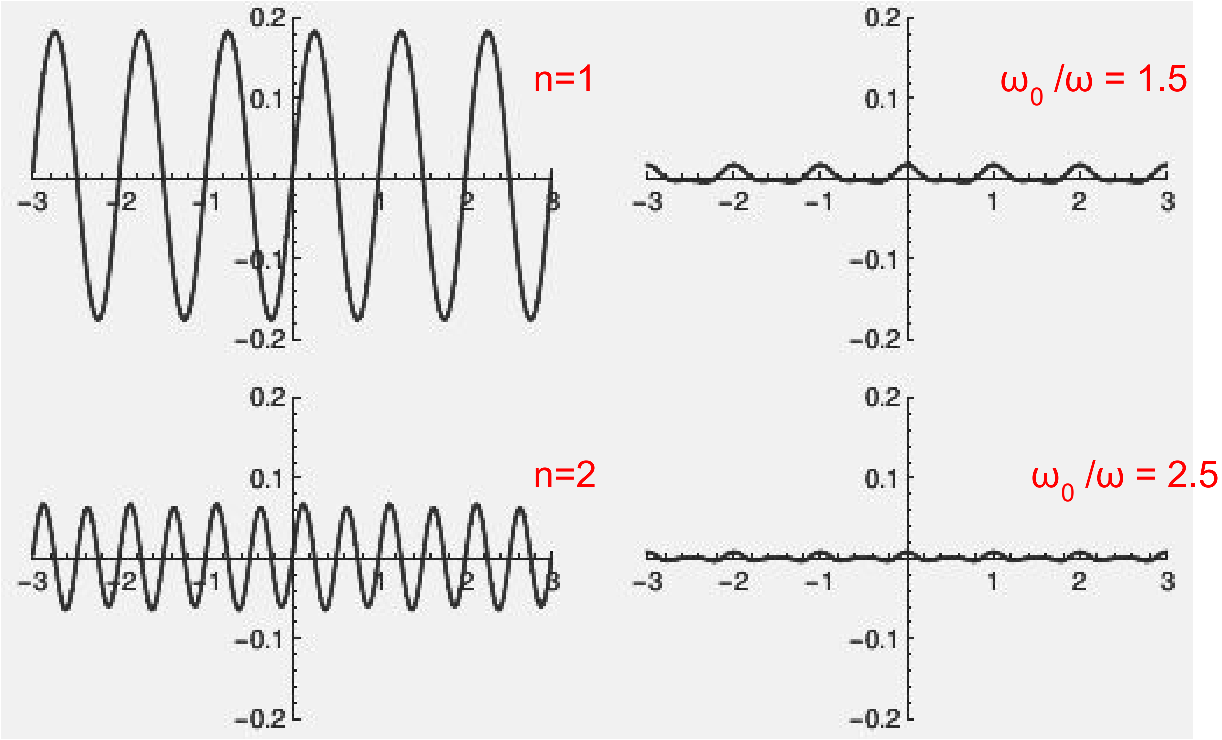
In Fig. 5.25, four values of τ are shown:

$\tau = 1.0\ \tau_0$;
$\tau = 1.5\ \tau_0$;
$\tau = 2.0\ \tau_0$;
$\tau = 2.5\ \tau_0$.

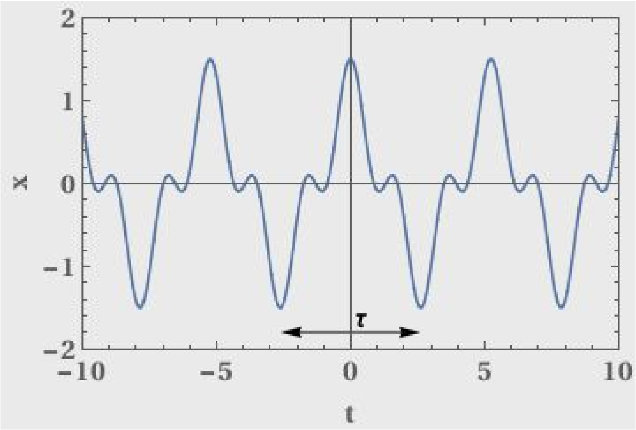
I.e.,
$\omega_0 / \omega = \tau / \tau_0 =$

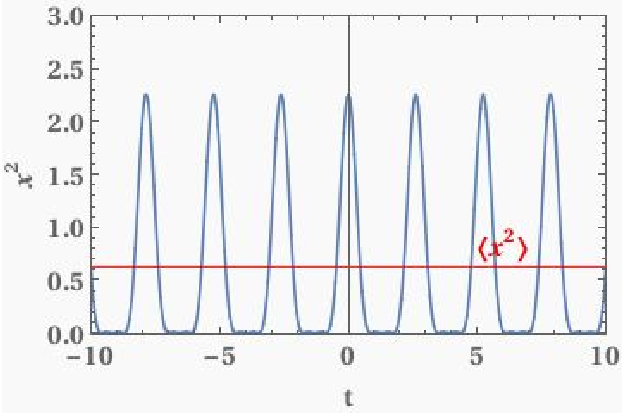
1
1.5
2
2.5

See Figure 5.25. Understand the resonance phenomenon: resonance occurs if
$n\omega = \omega_0$, for n = 1 2 3 …



Section 5.9. RMS displacement





- Given a periodic position x(t), with period τ and mean value 0, we define the RMS displacement by $x_{RMS} = \sqrt{\langle x^2 \rangle}$ where $\langle x^2 \rangle = 1/\tau \int_{-\tau/2}^{\tau/2} x(t)^2 \, dt$.
- RMS is **R**oot **M**ean **S**quare ; provides a quantitative measure of the displacements ;
- Parseval's theorem:

$$\langle x^2 \rangle = A_0^2 + \tfrac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)$$

The RMS displacement as a function of the drive period;

Figure 5.26 shows that resonance occurs at $n\omega = \omega_0$ for any integer n .

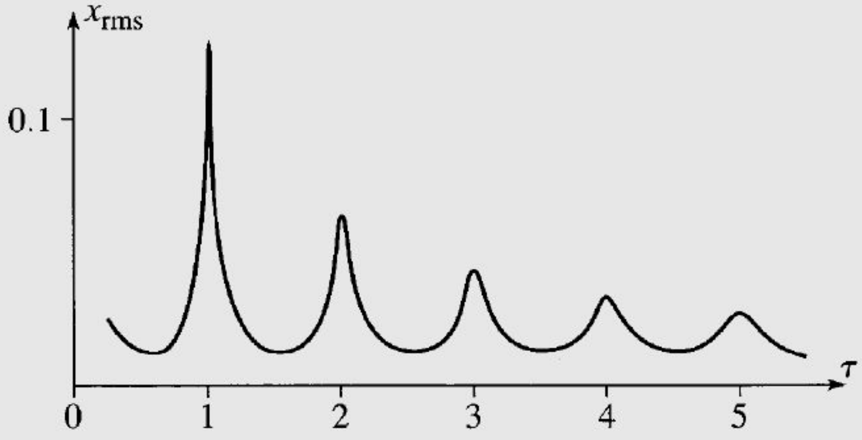


Figure 5.26 The RMS displacement of a linear oscillator, driven by periodic rectangular pulses, as a function of the drive period τ — calculated using the first six terms of the Parseval expression (5.100). The horizontal axis shows τ in units of the natural period τ_o. When τ is an integral multiple of τ_o the response is especially strong.

Homework Assignment #10
due in class Wednesday November 8
[47] Problem 4.53
[48] Problem 5.25 **
[49] Problem 5.30 **
[50] Problem 5.37 **
[51] Problem 5.44 **
[52] Problem 5.52 *** [Computer]

Use the cover sheet.

Exam 2 will be Friday November 3;

- conservation of energy;
- Section 4.6; "complete solution";
- central forces;
- damped oscillations;