Remaining …

Chapter 6.
Calculus of Variations

- *A topic in mathematics:*
  Find the function that minimizes an integral.
- *Solved by Leonhard Euler and Joseph-Louis Lagrange.*
- *Applies to a range of interesting problems.*

Chapter 7.
Lagrange's Equations

- *Lagrange developed a powerful method for deriving the equations of motion, which can be applied to generalized coordinates.*
- *It's related to the calculus of variations, by Hamilton's principle of least action*

Chapter 8. Motion with a Two-body Central Force

- *For example, the motion of the planets*

Chapter 6.
The Calculus of Variations

Read Chapter 6.

We'll spend only one week on Chapter 6.

THE VARIATIONAL PROBLEM

Consider a quantity S of this form,

$$S = \int_{x_1}^{x_2} f\left(y(x), y'(x), x \right) dx$$

where y(x) is a function whose values are specified at x_1 and x_2 ,

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2 \;;$$

also $y'(x) \equiv dy/dx$.

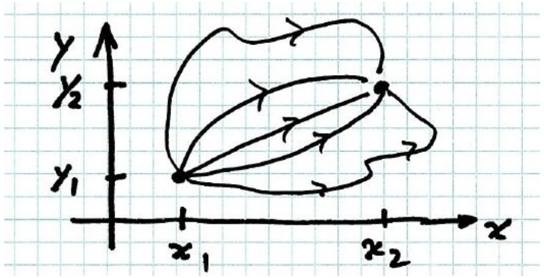
Terminology

S[y] is an example of a *functional*.

a function: $u \rightarrow g(u)$

a functional: $y(x) \rightarrow S[y]$

There are an infinite number of functions from (x_1,y_1) to (x_2,y_2) ;



and the value of S changes when we change the function.

The "variational problem" is to find the function y(x) for which S is minimum (or, maximum).

In the calculations below I'll assume that we seek the *minimum of S*; but the same equations apply for the *maximum*.

We say, "S is stationary".

$\delta S = 0.$

Let y(x) denote the function that makes S minimum,

$$S[\, y(x)\,] = \text{minimum value of } S;$$

then for any function(†) $\varepsilon(x)$,

$$S[\, y(x) + \varepsilon(x)\,] = S[\, y(x)\,] + \delta S$$

where $\delta S > 0$.

Now let $\varepsilon(x)$ be very small ("infinitesimal") and *calculate δS to linear accuracy in $\varepsilon(x)$.*

(†) *but we must keep the endpoints fixed;*

that is, $\varepsilon(x_1) = 0$ and $\varepsilon(x_2) = 0$.

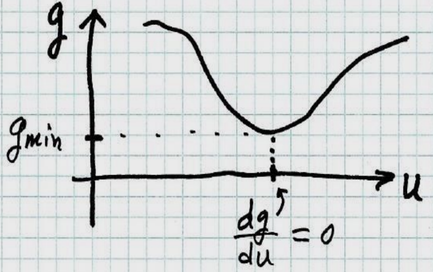
Define $\delta S = S[y + \varepsilon] - S[y]$

The condition for y(x) to be the function for which S has the minimum value, is that the *linear approximation of δS* must be equal to 0 for any $\varepsilon(x)$.

In other words, $\delta S = O(\varepsilon^2)$.

Or, $\delta S = 0$ to linear order .

Analogy: The minimum of a function g(u) occurs where $dg/du = 0$.



$g(u+\epsilon) = g(u) + \delta g$

$\delta g = \epsilon \, dg/du$ (small ϵ)

$\delta g = 0$ to linear order

$\Leftrightarrow dg/du = 0.$

The minimum of a functional S[y] occurs where

$$\frac{\delta S}{\delta y(x)} = 0 \qquad \text{for all x.}$$

Additional justification:
$\delta y(x) = \varepsilon(x)$; and so $\delta S/\delta y$ is the coefficient of the linearized approximation. If this coefficient is 0 then y(x) is at the minimum.

OK, now calculate δS to linear order…

$$\delta S = \int_{x_1}^{x_2} f[\,y+\epsilon,\ y'+\epsilon',\ x\,]\, dx - \int_{x_1}^{x_2} f(y, y', x)\, dx$$

& apply Taylor's theorem to linear order

$$\delta S = \int_{x_1}^{x_2} \left\{ f(y, y', x) + \epsilon \frac{\partial f}{\partial y} + \epsilon' \frac{\partial f}{\partial y'} \right\} dx - \int_{x_1}^{x_2} f(y, y', x)\, dx$$

$$\delta S = \int_{x_1}^{x_2} \left\{ \epsilon(x) \frac{\partial f}{\partial y} + \frac{d\epsilon}{dx} \frac{\partial f}{\partial y'} \right\} dx$$

Integration by parts

$$\frac{d\epsilon}{dx}\frac{\partial f}{\partial y'} = \underbrace{\frac{d}{dx}\left[\epsilon \frac{\partial f}{\partial y'}\right]} - \epsilon(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$\int_{x_1}^{x_2} (\text{''}) = 0$ because $= \epsilon \frac{\partial f}{\partial y'}\Big]_{x_1}^{x_2}$

$\epsilon(x_1) = \epsilon(x_2) = 0$

(fixed endpoints)

Thus

$$\delta S = \int_{x_1}^{x_2} \epsilon(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] dx$$

We demand $\delta S = 0$ for any $\epsilon(x)$.

Only way that can be true is if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Euler Lagrange equation

Result

Given the functional

$$S[y] = \int_{x1}^{x2} f\,(\,y(x)\,,\,y'(x)\,,\,x)\,dx$$

where $y(x_1) = y_1$ and $y(x_2) = y_2$ are fixed; the function $y(x)$ such that $S[y]$ is stationary obeys the *Euler-Lagrange equation*

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$$

<u>Preview of Chapter 7</u>

Calculus of Variations (Ch 6)

$$S = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

$$\delta S = 0 \quad \longleftrightarrow \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$$

In Chapter 7 we'll learn that the equation of motion for a mechanical system can be written as the Euler-Lagrange equation with

$$S = \int_{t_1}^{t_2} (T - U)\, dt$$

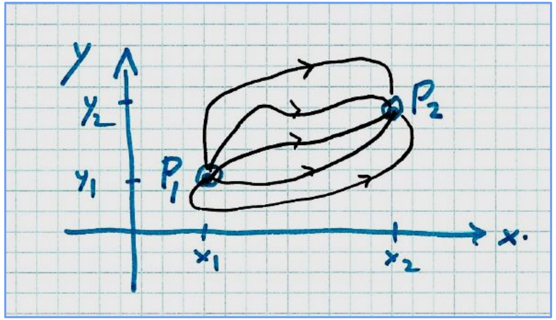
$T(\dot{x})$ $\quad$ $U(x)$ $\quad$ or use generalized coordinates

$\delta S = 0$ $\quad$ "principle of least action"

Example

the shortest distance between 2 points

Consider two points in 2 dimensions, $P_1 : (x_1,y_1)$ and $P_2 : (x_2,y_2)$.

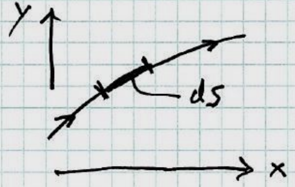


Use the Euler-Lagrange equations to determine the path from P_1 to P_2 that has the shortest distance.

(Of course you know the answer, but get it from the Eu.-Lagr. equation.)

Calculation

Arclength s ; $(ds)^2 = (dx)^2 + (dy)^2$



$$\Rightarrow \quad ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{1 + (dy/dx)^2}\; dx$$

$$\text{Length} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\; dx$$

$$f(y, y', x) = \sqrt{1 + (y')^2}$$

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{1}{2}\left[1 + (y')^2\right]^{-1/2} 2y'$$

"first integral"

E-L equation $\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$

Therefore $(\ldots)$ = constant ;

therefore y' = another constant $= m$

Solution $y(x) = mx + b$ where $y_1 = mx_1 + b$, $y_2 = mx_2 + b$

i.e., the straight line from P_1 to P_2. ✓

A couple of special cases

In general, $f(\,y(x)\,,\,y'(x)\,,\,x\,)$;

$\Longrightarrow$ we need to solve the differential equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \qquad (\bigstar)$$

Do you see that this is a second-order differential equation?

In two special cases we can reduce ($\bigstar$) to a *first-order diff. equation*, with an unknown constant that we can find from the initial conditions (**or other information**).

First special case : when f does not depend explicitly on y

$$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial y'} = \text{a constant}$$

$$\frac{\partial f}{\partial y'} = C \,, \quad \text{the ``first integral''}$$

TAYLOR PROBLEM 6.10

Second special case : when f does not depend explicitly on x

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad \frac{d}{dx}\left[f(y, y')\right]$$

$$= \underbrace{\frac{\partial f}{\partial y}}_{\to\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \underbrace{\frac{dy'}{dx}}_{\to\ y''}$$

$$= \frac{d}{dx}\left\{ \frac{\partial f}{\partial y'}\, y' \right\}$$

$$\frac{d}{dx}\left\{ f - y' \frac{\partial f}{\partial y'} \right\} = 0$$

TAYLOR PROBLEM 6.20

$$f - y'\frac{\partial f}{\partial y'} = C$$

the "first integral"

Other comments…

- ***Fermat's Principle*** is an application of Euler's equation in ***classical optics.***

- The ***Euler-Lagrange equation*** apply when we seek the stationary point of a functional. (A "point" in function space, means "a function".)

- ***Functional analysis*** in the path-integral form of quantum mechanics (R. P. Feynman) is based on exp{ i S / ħ } = the weighting of paths

Homework Assignment #11
due in class Wednesday November 15
[redacted]
[51] Problem 6.7 *
[52] Problem 6.8 *
[53] Problem 6.10 * and 6.20 **
[54] Problem 6.1* and 6.16 **
[55] Problem 6.19 **
[56] Problem 6.25 ***

Use the cover sheet.

Due Wednesday Nov. 8:
❊ ***Homework Assignment #10***