## Principal Definitions and Equations of Chapter 6

### The Euler–Lagrange Equation

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx \qquad \text{[Eq. (6.4)]}$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the **Euler–Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \qquad \text{[Eq. (6.13)]}$$

## Example 6.2.

*The brachistochrone*

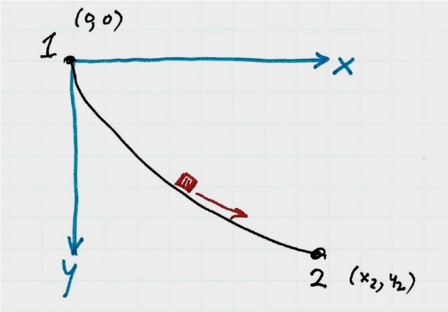
Read Chapter 6.

We'll spend only one week on Chapter 6.

The *brachistochrone problem* was posed by Johann Bernoulli in 1696. He sent a copy of the problem to Isaac Newton as a challenge; he thought maybe Newton wouldn't be able to solve it. Newton solved the problem overnight and sent the solution back to Bernoulli anonymously, as a kind of insult, to say "this is easy".

## THE BRACHISTOCHRONE

A small mass (ice cube, say) slides without friction down a curve from (0,0) to $(x_0,y_0)$.



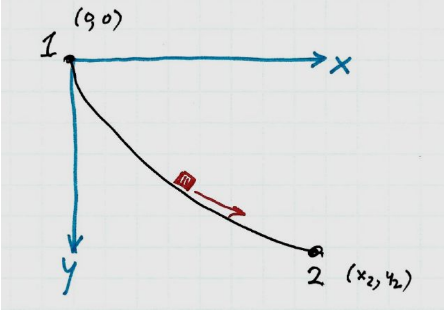

*What is the shape of the curve such that the ice cube slides to the bottom in the shortest time?*

"brachisto – chrone"

translates from Greek

as "shortest – time"

***– the curve of fastest descent –***

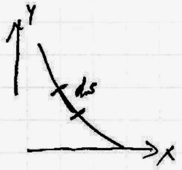



STEP I

we need a formula for the time of descent.

Using Taylor's notations

$$time(1\to 2) = \int_1^2 \frac{ds}{v}$$




$$ds = \sqrt{(dx)^2+(dy)^2}$$

$$= \sqrt{(dx/dy)^2+1}\; dy$$

$$v = \sqrt{2gy} \quad \text{by conservation of energy}$$

$$-mgy + \tfrac{1}{2}mv^2 = \text{constant}$$

$$0 = -mgy + \tfrac{1}{2}mv^2$$

$$v = \sqrt{2gy}$$

$$t_{12} = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{(dx/dy)^2+1}}{\sqrt{y}}\, dy$$

The function we need to determine is x(y).

*That requires we make some appropriate changes in the equations from last time:*

- *the independent variable today is "y" ;*
- *the dependent variable today is "x" ;*
- *the goal is to find x(y).*

$$t_{12} = \frac{1}{\sqrt{2g}} \int_0^{y_2} f(x,x',y)\, dy \qquad \text{where} \quad x' = \frac{dx}{dy}$$

$$f(x,x',y) = \frac{\sqrt{x'^2+1}}{\sqrt{y}}$$

*An important point is that the endpoints $(x_1,y_1)$ and $(x_2,y_2)$ are fixed.*

$$t_{12} = \int_{y_1}^{y_2} \underbrace{\frac{\sqrt{1+x'^2}}{\sqrt{y}}}_{f(x,x',y)}\, dy \quad /\sqrt{2g}$$

*Find the minimum of* $t_{12}$ .

**STEP II** Apply the Euler-Lagrange equation; *x is the function, y is the indep. variable ...*

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

$$0 = \frac{d}{dy}\left[\frac{1}{\sqrt{y}}\,\frac{1}{2}\left(1+x'^2\right)^{-\frac{1}{2}}\, 2x'\right]$$

$$= \frac{d}{dy}\left[\frac{x'}{\sqrt{y}\,\sqrt{(1+x'^2)}}\right]$$

**STEP III**

Solve the differential equation.

We already have a first integral, because f(x,x',y) does not depend on x! **[Problem 6.10]**

$$\frac{x'}{\sqrt{y}\,\sqrt{1+x'^2}} = \text{Constant} = \frac{1}{\sqrt{2a}}$$

***Call the constant 1 /√(2a), and interpret " a " later.***

$$\frac{(x')^2}{y} = (1+x'^2)\,\frac{1}{2a} \Rightarrow (x')^2\left(\frac{1}{y} - \frac{1}{2a}\right) = \frac{1}{2a}$$

$$x' = \frac{dx}{dy} = \sqrt{\frac{y}{2a-y}}$$

This we can solve by direct integration.

*Do the indefinite integral;*
*put in the end points later.*

Integration

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

Change the variable of integration from $y$ to $\theta$, related by $y = a(1-\cos\theta)$

$$x = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\, a\sin\theta\, d\theta$$

$$= \int (1-\cos\theta)\, a\, d\theta$$

$$= a(\theta - \sin\theta)$$

$$x = a(\theta - \sin\theta)$$

$$y = a(1-\cos\theta)$$

when $\theta$ goes from $0$ $\left( (x_1, y_1) = (0,0) \right)$ to $\theta_2$ where $\begin{cases} x_2 = a(\theta_2 - \sin\theta_2) \\ y_2 = a(1-\cos\theta_2) \end{cases}$

Note that the boundary $(x_2, y_2)$ determines the constants $(a, \theta_2)$.

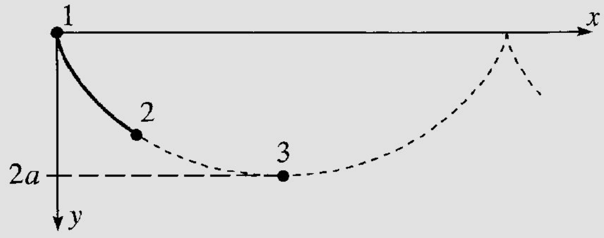



Figure 6.5 The path for a roller coaster that gives the shortest time between the given points 1 and 2 is part of the cycloid with a

⇒ Parametric Equations for the Brachistochrone Curve

***Figure 6.5***

Final result,

***The brachistochrone is a segment of a cycloid curve.*** [Bernoulli ; Newton]

*Parametric equations*

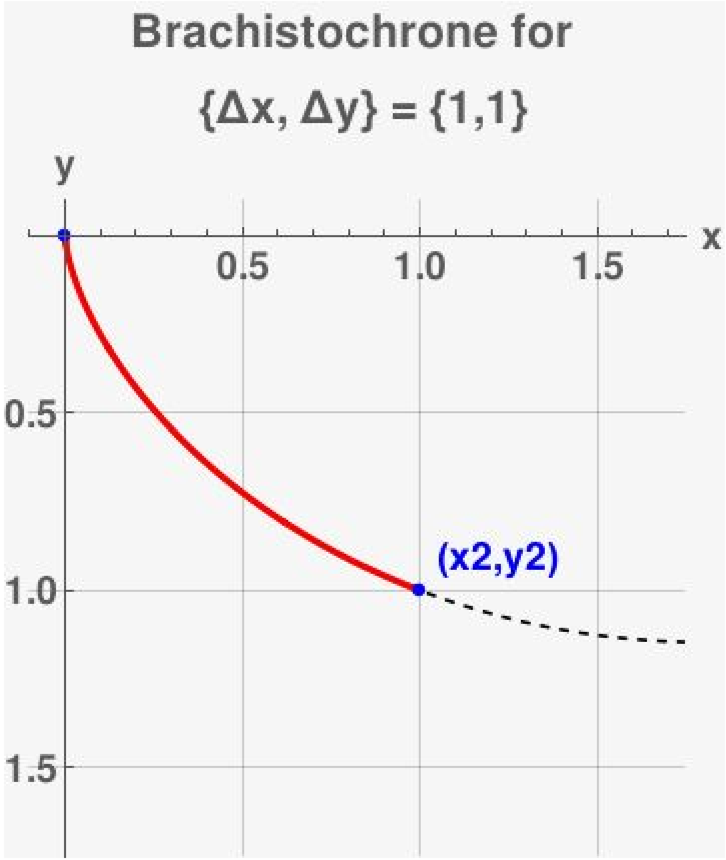
$$x(\theta) = a\,(\,\theta - \sin\theta\,)$$

$$y(\theta) = a\,(\,1 - \cos\theta\,)$$

The answer depends on two constants ($a$ and $\theta_2$ ) which are determined from the coordinates of the final point ($x_2$ and $y_2$ ) :

$$x_2 = a\,(\theta_2 - \sin\theta_2)$$

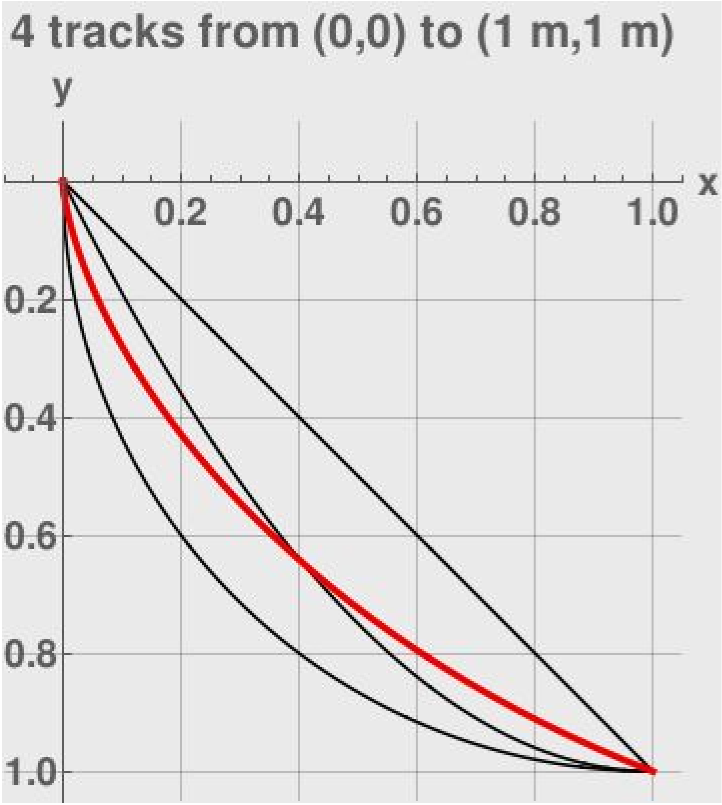
$$y_2 = a\,(1 - \cos\theta_2)$$



## Compare different curves

The high point is $(x_1, y_1) = (0, 0)$;
the low point is $(x_2, y_2) = (1\text{ m}, 1\text{ m})$.
($g = 9.8\text{ m/s}^2$)

| shape of the track | time (0,0) −> (1 m, 1 m) |
|---|---|
| straight line | 0.6389 s |
| parabola | 0.5952 s |
| circular arc | 0.5923 s |
| brachistochrone | 0.5832 s |

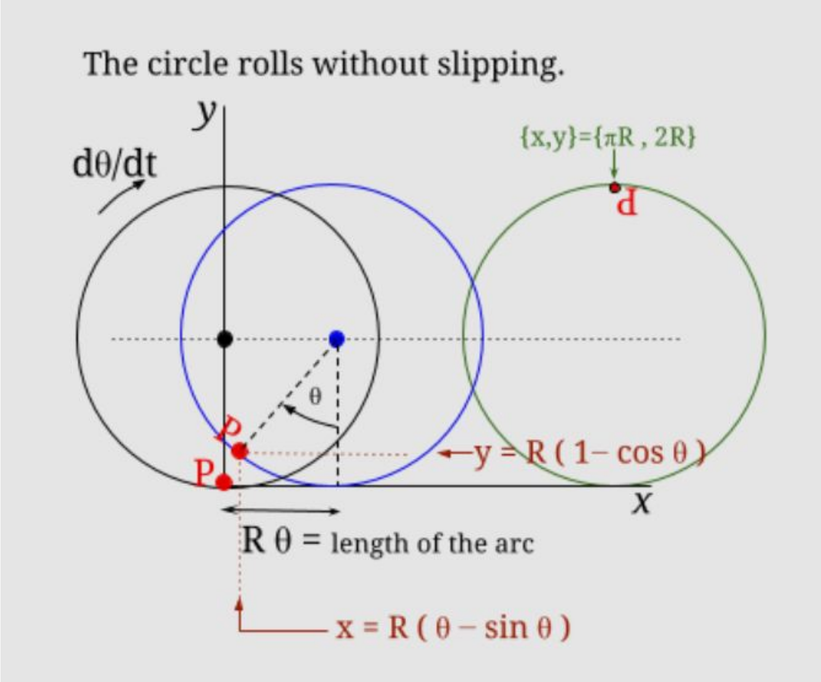


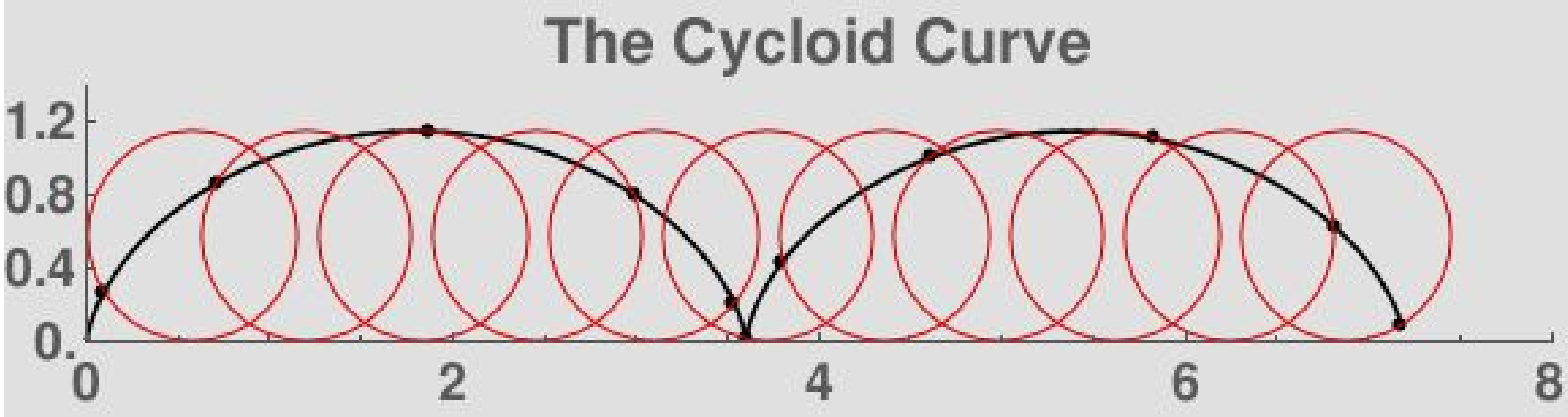
## Mathematics of the Cycloid Curve

A circle (radius = $R$) rolls without slipping on the x axis.

What is the curve traced out by P, a point on the rolling circle?

$$x(\theta) = R\,(\,\theta - \sin\theta\,)$$
$$y(\theta) = R\,(\,1 - \cos\theta\,)$$




The Cycloid Curve
1.2
0.8
0.4
0.
0
2
4
6
8

## A related problem

**The tautochrone problem –**

– identify the curve such that the time of descent to the bottom (P) is the same for any initial point ($P_0$);

– solved by Christiaan Huygens. He proved, in his book *Horologium Oscillatorium*, published in 1673, that the curve is a cycloid.

= Taylor Problem 6.25.

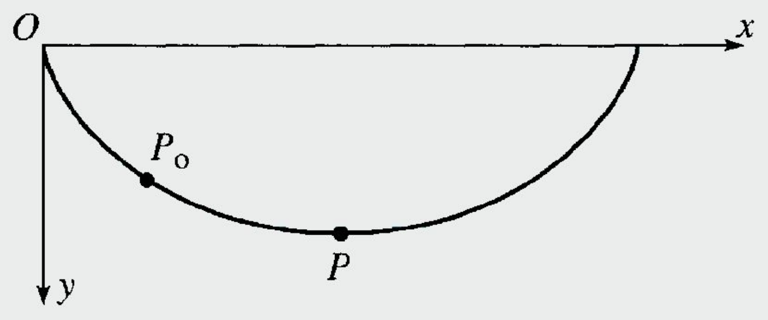



Figure 6.11 Problem 6.25

Homework Assignment #11
due in class Wednesday November 15
[51] Prob. 6.7*
[52] Prob. 6.8*
[53] Probs. 6.10* and 6.20**
[54] Probs. 6.1* and 6.16**
[55] Prob. 6.19**
[56] Prob. 6.25***
***Use the cover sheet.***