Principal Definitions and Equations of Chapter 6

The Euler–Lagrange Equation

An integral of the form

\[ S = \int_{x_1}^{x_2} f[y(x), y'(x), x] \, dx \]  \hspace{1cm} \text{[Eq. (6.4)]}

taken along a path \( y = y(x) \) is stationary with respect to variations of that path if and only if \( y(x) \) satisfies the Euler–Lagrange equation

\[ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0. \]  \hspace{1cm} \text{[Eq. (6.13)]}
Example 6.2.  

*The brachistochrone*

Read Chapter 6.

We'll spend only one week on Chapter 6.

The *brachistochrone problem* was posed by Johann Bernoulli in 1696. He sent a copy of the problem to Isaac Newton as a challenge; he thought maybe Newton wouldn't be able to solve it. Newton solved the problem overnight and sent the solution back to Bernoulli anonymously, as a kind of insult, to say "this is easy".

**THE BRACHISTOCHRONE**

A small mass (ice cube, say) slides without friction down a curve from (0,0) to $(x_0, y_0)$.

What is the shape of the curve such that the ice cube slides to the bottom in the shortest time?

"brachisto – chrone"

translates from Greek as "shortest – time"

– the curve of fastest descent –
The function we need to determine is \( x(y) \).

That requires we make some appropriate changes in the equations from last time:
- the independent variable today is "y";
- the dependent variable today is "x";
- the goal is to find \( x(y) \).

An important point is that the endpoints \((x_1, y_1)\) and \((x_2, y_2)\) are fixed.

**STEP I**
we need a formula for the time of descent.

Using Taylor's notations

\[
\begin{align*}
t_{12} &= \frac{1}{\sqrt{2g}} \int_0^{y_2} f(x, x', y) \, dy \\
&= \frac{1}{\sqrt{2g}} \int_0^{y_2} \sqrt{(dx/dy)^2 + 1} \, dy \\
f(x, x', y) &= \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}
\end{align*}
\]
STEP II  Apply the Euler-Lagrange equation; \( x \) is the function, \( y \) is the indep. variable ...

\[
\frac{\partial f}{\partial x} = \frac{d}{dy} \left( \frac{\partial f}{\partial x'} \right)
\]

\[
0 = \frac{d}{dy} \left[ \frac{1}{\sqrt{y}} \frac{1}{2} (1 + x'^2)^{-\frac{1}{2}} 2x' \right]
\]

\[
= \frac{d}{dy} \left[ \frac{x'}{\sqrt{y} \sqrt{1 + x'^2}} \right]
\]

STEP III  Solve the differential equation.

We already have a first integral, because \( f(x, x', y) \) does not depend on \( x \)! [Problem 6.10]

\[
\frac{x'}{\sqrt{y} \sqrt{1 + x'^2}} = \text{constant} = \frac{1}{\sqrt{2a}}
\]

Call the constant \( 1 / \sqrt{2a} \), and interpret "a" later.

\[
\frac{(x')^2}{y} = \frac{1}{2a} (1 + x'^2) \Rightarrow \alpha y^2 \left( \frac{1}{y} - \frac{1}{2a} \right) = \frac{1}{2a}
\]

\[
x' = \frac{dx}{dy} = \sqrt{\frac{y}{2a - y}}
\]

This we can solve by direct integration.
Do the indefinite integral; put in the end points later.

\[ x = \int \sqrt{\frac{y}{2a-y}} \, dy \]

Change the variable of integration from \( y \) to \( \theta \), related by \( y = a(1-\cos \theta) \)

\[ x = \int \frac{\sqrt{1-\cos \theta}}{1+\cos \theta} \sin \theta \, d\theta \]

\[ = \int (1-\cos \theta) \, d\theta \]

\[ = a(\theta - \sin \theta) \]

⇒ Parametric Equations for the Brachistochrone Curve

**Figure 6.5**

\[ x = a(\theta - \sin \theta) \]

\[ y = a(1-\cos \theta) \]

When \( \theta \) goes from 0 (\( (x_1, y_1) = (0, 0) \)) to \( \theta_2 \) where \( \int x_2 = a(\theta_2 - \sin \theta_2) \)

\[ \int y_2 = a(1-\cos \theta_2) \]

Note that the boundary \((x_0, y_2)\) determines the constants \((a, \theta_2)\).

Figure 6.5 The path for a roller coaster that gives the shortest time between the given points 1 and 2 is part of the cycloid with a
Final result,  
*The brachistochrone is a segment of a cycloid curve.* [Bernoulli; Newton]

**Parametric equations**

\[
\begin{align*}
x(\theta) &= a ( \theta - \sin \theta ) \\
y(\theta) &= a ( 1 - \cos \theta )
\end{align*}
\]

The answer depends on two constants \((a \text{ and } \theta_2)\) which are determined from the coordinates of the final point \((x_2 \text{ and } y_2)\):

\[
\begin{align*}
x_2 &= a ( \theta_2 - \sin \theta_2 ) \\
y_2 &= a ( 1 - \cos \theta_2 )
\end{align*}
\]
Compare different curves

The high point is \((x_1, y_1) = (0, 0)\); the low point is \((x_2, y_2) = (1 \, \text{m}, 1 \, \text{m})\). \((g = 9.8 \, \text{m/s}^2)\)

<table>
<thead>
<tr>
<th>shape of the track</th>
<th>time ((0,0) \rightarrow (1 , \text{m}, 1 , \text{m}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>straight line</td>
<td>0.6389 s</td>
</tr>
<tr>
<td>parabola</td>
<td>0.5952 s</td>
</tr>
<tr>
<td>circular arc</td>
<td>0.5923 s</td>
</tr>
<tr>
<td>brachistochrone</td>
<td>0.5832 s</td>
</tr>
</tbody>
</table>
Mathematics of the Cycloid Curve

A circle (radius = \( R \)) rolls without slipping on the x axis.

What is the curve traced out by \( P \), a point on the rolling circle?

\[
\begin{align*}
x(\theta) &= R \left( \theta - \sin \theta \right) \\
y(\theta) &= R \left( 1 - \cos \theta \right)
\end{align*}
\]
The Cycloid Curve
A related problem

The tautochrone problem –

– identify the curve such that the time of descent to the bottom (P) is the same for any initial point (P₀);

– solved by Christiaan Huygens. He proved, in his book *Horologium Oscillatorium*, published in 1673, that the curve is a cycloid.

= Taylor Problem 6.25.

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Homework Assignment #11
due in class Wednesday November 15
[51] Prob. 6.7*
[52] Prob. 6.8*
[53] Probs. 6.10* and 6.20**
[54] Probs. 6.1* and 6.16**
[55] Prob. 6.19**
[56] Prob. 6.25***

*Use the cover sheet.*