Principal Definitions and Equations of Chapter 6

The Euler–Lagrange Equation

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx \qquad \text{[Eq. (6.4)]}$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the **Euler–Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0. \qquad \text{[Eq. (6.13)]}$$

Examples

EXAMPLE 6.1

- the shortest path between two points on a plane

EXAMPLE 6.2

- the brachistochrone

EXAMPLE 6.3

- the shortest path between two points on a plane, using more than two variables.

Homework Assignment #11
due Wednesday November 15

[51] Prob. 6.7*
[52] Prob. 6.8*
[53] Probs. 6.10* and 6.20**
[54] Probs. 6.1* and 6.16**
[55] Prob. 6.19**
[56] Prob. 6.25***

Use the cover sheet.

Examples

PROBLEM 6.7 [51]

- Find the geodesic on a cylinder.

PROBLEM 6.8 [52]

- Show that a roller coaster car has v = SQRT(2gy).

PROBLEM 6.9

- Find the curve that makes

$$\int_0^P \left(y'^2 + y\,y' + y^2 \right) dx$$

stationary.

Examples

PROBLEM 6.10 [53]

- Find the "*first integral*," if f(y,y',x) is independent of y.

PROBLEM 6.11

- Find the curve that makes

$$\int_{x1}^{x2} \mathrm{SQRT}[\; x\,(1+y'^2)\,]\; dx$$

stationary.

PROBLEM 6.12

- Find the curve that makes

$$\int_{x1}^{x2} x\;\mathrm{SQRT}[\,(1+y'^2)\,]\; dx$$

stationary.

Examples

PROBLEM 6.13

- Find the geodesic on a sphere.

PROBLEM 6.14

- Show that the brachistochrone is a section of a cycloid curve.

PROBLEM 6.15

- Solve the brachistochrone problem with an initial speed.

Examples

PROBLEM 6.16 [54]

- Find the geodesic on a sphere.

PROBLEM 6.17

- Find the geodesic on a cone.

PROBLEM 6.18

- Find the shortest path on a plane, using polar coordinates.

Examples

PROBLEM 6.19 [55]

- The soap bubble problem

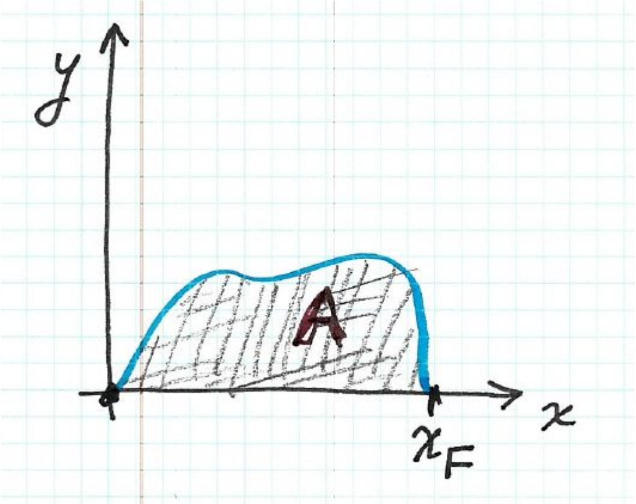
PROBLEM 6.20

- Find the "*first integral*," if f(y,y',x) is independent of x.

PROBLEM 6.21

- Solve the brachistochrone problem again.

PROBLEM 6.22



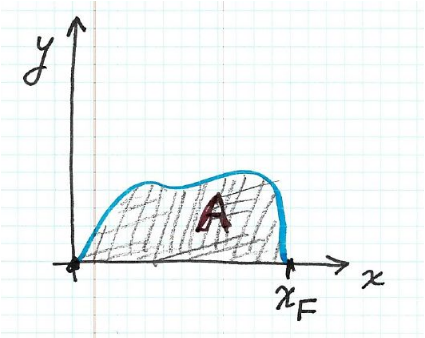
Taylor Problem 6.22 ***

Consider a flexible string with fixed length = l .

One end of the string is pinned at the origin (0,0) in the xy-plane. The other end can be pinned at any point on the x axis. Then the string forms a curve in the xy-plane.

Determine the curve for which the area A is maximum.

(You can probably guess the answer, but can you *prove* it?)



$$A = \int_0^{x_F} y\, dx$$

But this does not have the right form, because the endpoints are not fixed.

Let s = arclength and describe the curve by *y(s)*.

Now the endpoints are fixed , because *y(0) = 0* and *y(l) = 0*.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$dx = \sqrt{(ds)^2 - (dy)^2} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2}\, ds$$

$$A = \int_0^{l} y(s)\sqrt{1-(y')^2}\, ds \quad \leftarrow y' = \frac{dy}{ds}$$

$$f(y, y', s) = y\sqrt{1-(y')^2}$$

$$\frac{\partial f}{\partial y} = \sqrt{1-(y')^2} \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{-yy'}{\sqrt{1-(y')^2}}$$

The Euler Lagrange equation is

$$\sqrt{1-(y')^2} = \frac{d}{ds}\left[\frac{-yy'}{\sqrt{1-(y')^2}}\right]$$

We already know the first integral, because $\frac{\partial f}{\partial s} = 0$.

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

Verify:

$$\frac{d}{ds}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' - y''\frac{\partial f}{\partial y'} - y'\frac{d}{ds}\left(\frac{\partial f}{\partial y'}\right)$$

$$= y'\left[\frac{\partial f}{\partial y} - \frac{d}{ds}\left(\frac{\partial f}{\partial y'}\right)\right] = 0 \quad ✓$$

$$y\sqrt{1-(y')^2} - \frac{-y(y')^2}{\sqrt{1-(y')^2}} = C$$

$$= \frac{y}{\sqrt{1-(y')^2}}\left[1-(y')^2+(y')^2\right] = \frac{y}{\sqrt{1-(y')^2}}$$

To solve : $y = C\sqrt{1-\left(\frac{dy}{ds}\right)^2}$

We can solve it by a change of variables.

Let $y = R\sin\theta$ — R = some constant, θ = the new variable

$$\frac{dy}{ds} = R\cos\theta\,\frac{d\theta}{ds}$$

$$R\sin\theta = C\sqrt{1-R^2\cos^2\theta\left(\frac{d\theta}{ds}\right)^2}$$

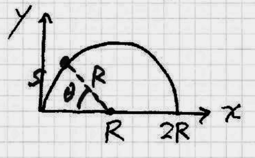
The solution is $R^2\left(\frac{d\theta}{ds}\right)^2 = 1$ ← $\theta = s/R$

which gives $R\sin\theta = C\sqrt{1-\cos^2\theta}$
$= C\sin\theta$, so $R = C$.

Result $y(s) = R\sin\left(s/R\right)$ and $R = C$.

(So far, R is an unknown constant.)

The result is the equation for a half circle, in terms of arclength s

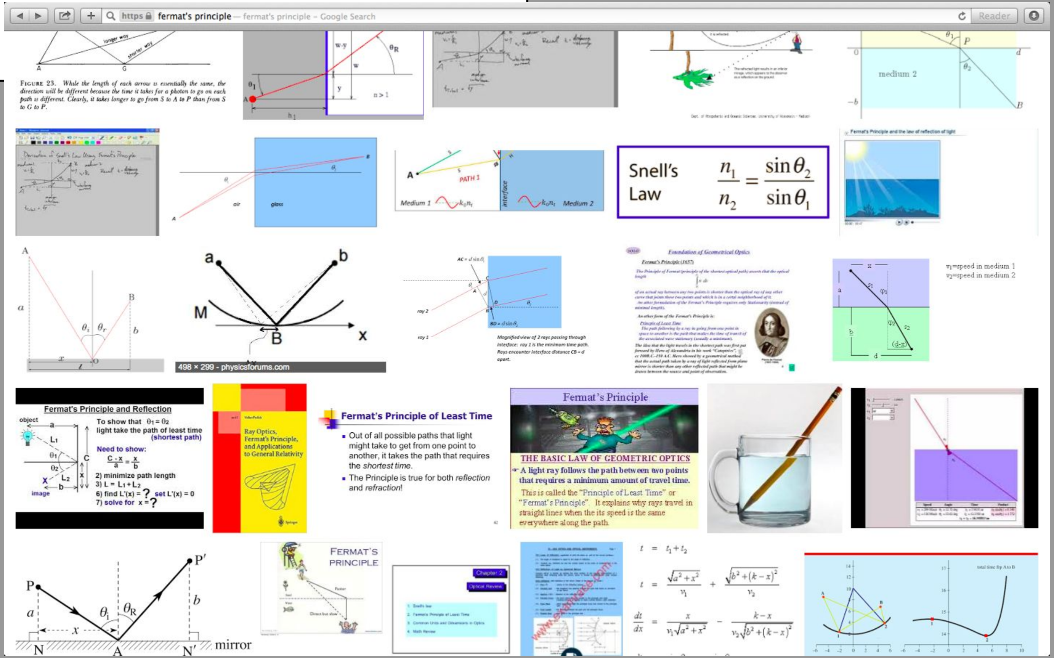


$$\left.\begin{aligned} x &= R - R\cos\theta \\ y &= R\sin\theta \\ s(\theta) &= R\theta \end{aligned}\right\} \quad y = R\sin\left(\frac{s}{R}\right) ✔$$

But now, what is R? $\pi R = \ell$.

PROBLEM 6.23
Direct an aircraft flying long distance through a ***wind shear*** .

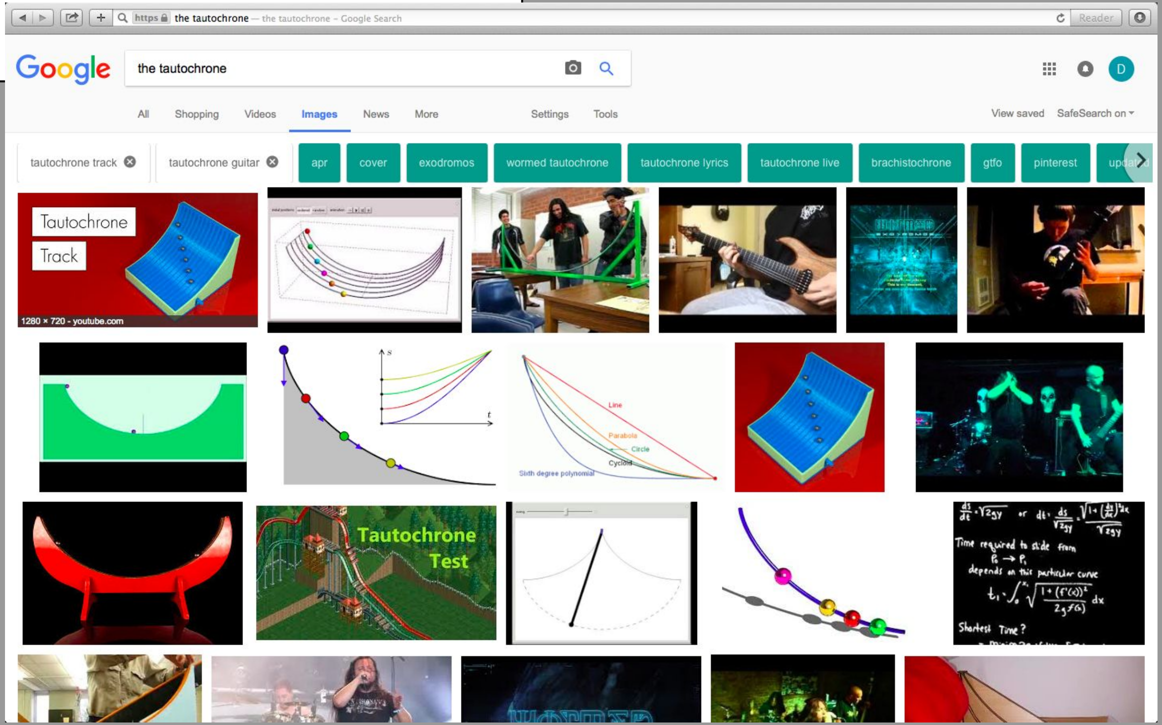
PROBLEM 6.24
Solve an optics problem using Fermat's principle.



Examples

PROBLEM 6.25   [56]

❑ The tautochrone problem



Quiz Question

Define $S = \int_{t0}^{t1} \{ \frac{1}{2} m (dx/dt)^2 - U(x(t)) \} dt$

where $x(t_0) = x_0$ and $x(t_1) = x_1$.

Derive the Euler-Lagrange equation, for the path from (t_0 , x_0) to (t_1 , x_1) with the minimum value of S.

In mechanics, S is called the "action."