

Principal Definitions and Equations of Chapter 6

The Euler–Lagrange Equation

An integral of the form

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x] dx \quad [\text{Eq. (6.4)}]$$

taken along a path $y = y(x)$ is stationary with respect to variations of that path if and only if $y(x)$ satisfies the **Euler–Lagrange equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0. \quad [\text{Eq. (6.13)}]$$

Examples

EXAMPLE 6.1

- the shortest path between two points on a plane

EXAMPLE 6.2

- the brachistochrone

EXAMPLE 6.3

- the shortest path between two points on a plane, using more than two variables.

Homework Assignment #11

due Wednesday November 15

[51] Prob. 6.7*

[52] Prob. 6.8*

[53] Probs. 6.10* and 6.20**

[54] Probs. 6.1* and 6.16**

[55] Prob. 6.19**

[56] Prob. 6.25***

Use the cover sheet.

Examples

PROBLEM 6.7 [51]

- Find the geodesic on a cylinder.

PROBLEM 6.8 [52]

- Show that a roller coaster car has $v = \text{SQRT}(2gy)$.

PROBLEM 6.9

- Find the curve that makes

$$\int_0^P (y'^2 + y y' + y^2) dx$$

stationary.

Examples

PROBLEM 6.10 [53]

- Find the "*first integral*," if $f(y, y', x)$ is independent of y .

PROBLEM 6.11

- Find the curve that makes

$$\int_{x_1}^{x_2} \text{SQRT}[x(1+y'^2)] dx$$

stationary.

PROBLEM 6.12

- Find the curve that makes

$$\int_{x_1}^{x_2} x \text{SQRT}[1+y'^2] dx$$

stationary.

Examples

PROBLEM 6.13

- ❑ Find the geodesic on a sphere.

PROBLEM 6.14

- ❑ Show that the brachistochrone is a section of a cycloid curve.

PROBLEM 6.15

- ❑ Solve the brachistochrone problem with an initial speed.

Examples

PROBLEM 6.16 [54]

- ❑ Find the geodesic on a sphere.

PROBLEM 6.17

- ❑ Find the geodesic on a cone.

PROBLEM 6.18

- ❑ Find the shortest path on a plane, using polar coordinates.

Examples

PROBLEM 6.19 [55]

- ❑ The soap bubble problem

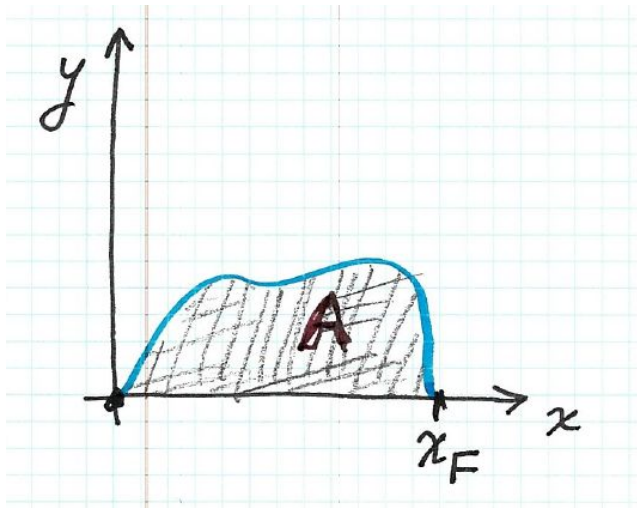
PROBLEM 6.20

- ❑ Find the "*first integral*," if $f(y,y',x)$ is independent of x .

PROBLEM 6.21

- ❑ Solve the brachistochrone problem again.

PROBLEM 6.22



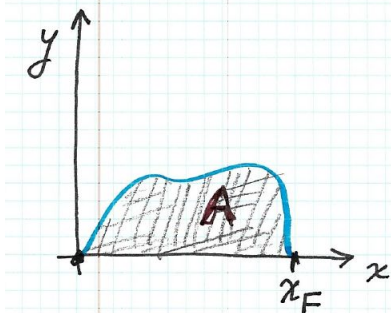
Taylor Problem 6.22 ***

Consider a flexible string
with fixed length = 1 .

One end of the string is pinned at the origin $(0,0)$ in the xy -plane. The other end can be pinned at any point on the x axis. Then the string forms a curve in the xy -plane.

Determine the curve for which the area A is maximum.

(You can probably guess the answer, but can you *prove* it?)



$$A = \int_0^{x_F} y \, dx$$

But this does not have the right form, because the endpoints are not fixed.

Let s = arclength and describe the curve by $y(s)$.

Now the endpoints are fixed, because $y(0) = 0$ and $y(l) = 0$.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$dx = \sqrt{(ds)^2 - (dy)^2} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2} ds$$

$$A = \int_0^l y(s) \sqrt{1 - (y')^2} ds \quad \leftarrow y' = \frac{dy}{ds}$$

$$f(y, y', s) = y \sqrt{1 - (y')^2}$$

$$\frac{\partial f}{\partial y} = \sqrt{1 - (y')^2} \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{-yy'}{\sqrt{1 - (y')^2}}$$

The Euler Lagrange equation is

$$\sqrt{1 - (y')^2} = \frac{d}{ds} \left[\frac{-yy'}{\sqrt{1 - (y')^2}} \right]$$

We already know the first integral, because $\frac{\partial f}{\partial s} = 0$.

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} = C$$

Verify:

$$\begin{aligned} \frac{d}{ds} \left[f - y' \frac{\partial f}{\partial y'} \right] &= \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' - y'' \frac{\partial f}{\partial y'} - y' \frac{d}{ds} \left(\frac{\partial f}{\partial y'} \right) \\ &= y' \left[\frac{\partial f}{\partial y} - \frac{d}{ds} \left(\frac{\partial f}{\partial y'} \right) \right] = 0 \quad \checkmark \end{aligned}$$

$$y\sqrt{1-(y')^2} - \frac{-y(y')^2}{\sqrt{1-(y')^2}} = c$$

$$= \frac{y}{\sqrt{1-(y')^2}} [1-(y')^2 + (y')^2] = \frac{y}{\sqrt{1-(y')^2}}$$

To solve : $y = c \sqrt{1 - \left(\frac{dy}{ds}\right)^2}$

We can solve it by a change of variables.

Let $y = R \sin \theta$ $R = \text{some constant}$
 $\theta = \text{the new variable}$

$$\frac{dy}{ds} = R \cos \theta \frac{d\theta}{ds}$$

$$R \sin \theta = c \sqrt{1 - R^2 \cos^2 \theta \left(\frac{d\theta}{ds}\right)^2}$$

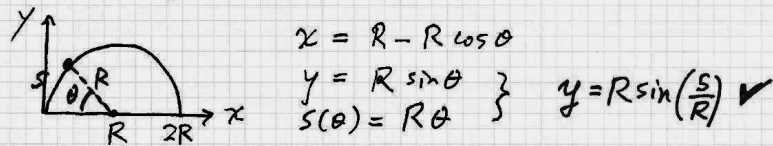
The solution is $R^2 \left(\frac{d\theta}{ds}\right)^2 = 1$ $\leftarrow \theta = s/R$

which gives $R \sin \theta = c \sqrt{1 - \cos^2 \theta}$
 $= c \sin \theta$, so $R = c$.

Result $y(s) = R \sin\left(\frac{s}{R}\right)$ and $R = c$.

(So far, R is an unknown constant.)

The result is the equation for a half circle, in terms of arclength s



But now, what is R? $\pi R = l$.

PROBLEM 6.23

Direct an aircraft flying long distance through a **wind shear**.

PROBLEM 6.24

Solve an optics problem using Fermat's principle.

fermat's principle - Google Search

FIGURE 23 While the length of each arrow is essentially the same, the direction will be different because the time it takes for a photon to go on each path is different. Clearly, it takes longer to go from S to A to P than from S to C to P.

Snell's Law:
$$\frac{n_1}{n_2} = \frac{\sin \theta_2}{\sin \theta_1}$$

Fermat's Principle and the Law of Reflection of Light

Fermat's Principle (LAST)
The principle of least principle of the shortest optical path means that the optical path length between two points is a minimum. This is a consequence of Fermat's Principle which states that light travels between two points along the path that requires the least time.

Fermat's Principle of Least Time

To show that $\theta_1 = \theta_2$ light take the path of least time (shortest path)

Need to show:

$$\frac{dL}{dx} = 0$$

$$L = L_1 + L_2$$

$$\text{Find } L(x) = ? \text{ and } L'(x) = 0$$

$$\text{Solve for } x = ?$$

Fermat's Principle

THE BASIC LAW OF GEOMETRIC OPTICS
 • A light ray follows the path between two points that requires a minimum amount of travel time.
 This is called the "Principle of Least Time" or "Fermat's Principle". It explains why rays travel in straight lines when the air speed is the same everywhere along the path.

$$L = L_1 + L_2$$

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (k-x)^2}$$

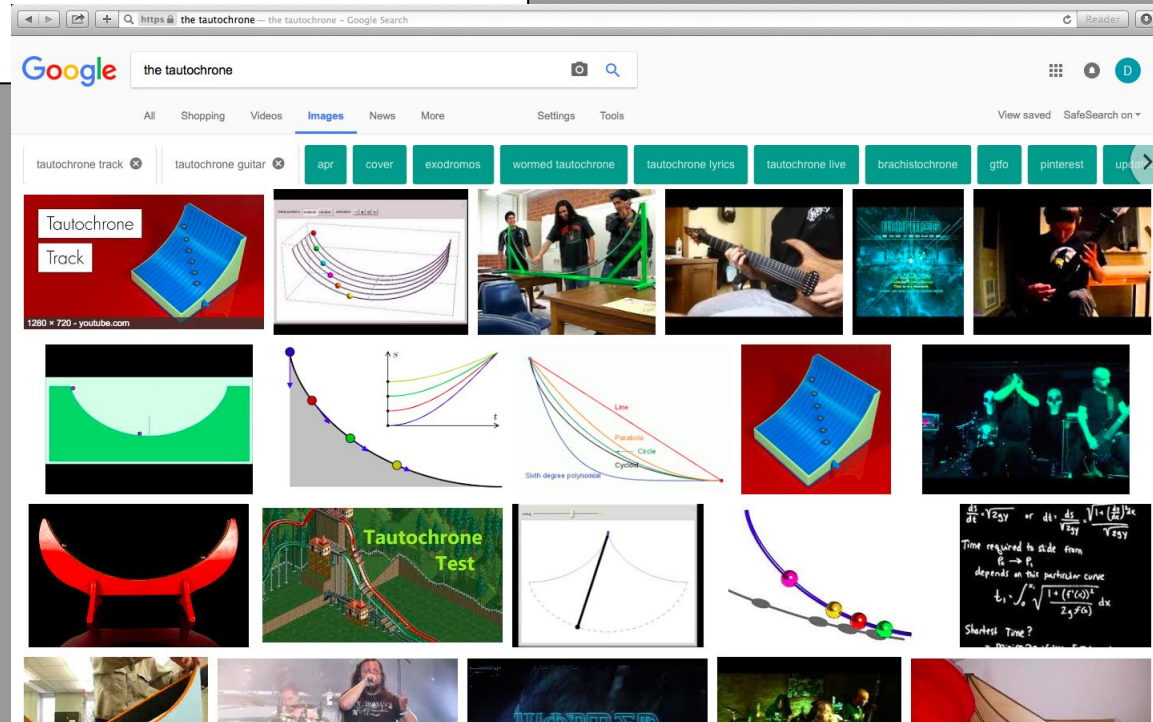
$$\frac{dL}{dx} = \frac{x}{\sqrt{a^2 + x^2}} - \frac{k-x}{\sqrt{b^2 + (k-x)^2}}$$

Set $\frac{dL}{dx} = 0$ and solve for x .

Examples

PROBLEM 6.25 [56]

☐ The tautochrone problem

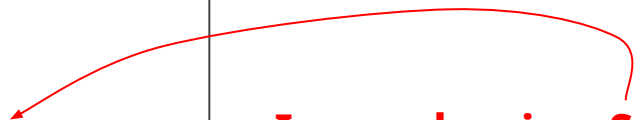


Quiz Question

Define $S = \int_{t_0}^{t_1} \left\{ \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - U(x(t)) \right\} dt$

where $x(t_0) = x_0$ and $x(t_1) = x_1$.

Derive the Euler-Lagrange equation, for the path from (t_0, x_0) to (t_1, x_1) with the minimum value of S .



In mechanics, S is called the "action."