FRIDAY'S QUIZ QUESTION

AN EXAMPLE IN THE CALCULUS OF VARIATIONS …

GIVEN $S = \int [\, \tfrac{1}{2} m \dot{x}^2 - U(x) \,]\, dt$,
find the x(t) that makes S[x(t)] stationary.

<u>SOLUTION</u> It's an Euler problem, so the solution is given by Euler's equation,

$$\frac{\partial f}{\partial x} = \frac{d}{dt} \frac{\partial f}{\partial \dot{x}}$$

where $f(x, \dot{x}, t) = \tfrac{1}{2} m \dot{x}^2 - U(x)$.

You should recognize the Euler-Lagrange equation here.

That is,

$$-\frac{\partial U}{\partial x} = \frac{d}{dt} m \dot{x}$$

$$m \ddot{x} = -\partial U / \partial x$$

Chapter 7. Lagrange's Equations

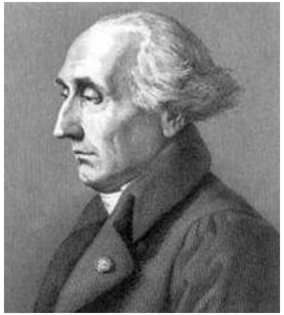
1 Historical Introduction

Joseph-Louis Lagrange

(1736-1813)

Berlin; Paris;

<u>*Mécanique analytique*</u>



William Rowan Hamilton

(1805-1865)

Dublin;

"*On a General Method in Dynamics*"

Section 7.1. Lagrange's Equations for *Unconstrained* Motion

What do we mean by "unconstrained" motion?

The particle moves in 3 dimensions under the influence of a conservative net force.

The potential energy is U(**r**).

$$T = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

$$U = U(\mathbf{r})$$

The *Lagrangian* is $\mathcal{L} = T - U$.

(Notation: Script L)

(An example of "constrained motion" would be something like curvilinear motion of a bead on a wire, or planar motion.)

2 Lagrange's equations

We define $\pounds = T - U$.

Think of this as a function of $\{x,y,z\}$ and $\{\dot{x}, \dot{y}, \dot{z}\}$; i.e.,

$$\pounds = \pounds(\mathbf{r}, \dot{\mathbf{r}}) \qquad \color{red}{\mathbf{r} = \{x,y,z\}}$$

Now, *understand the partial derivative* :

$\partial / \partial x$ means

vary x but keep the other 5 variables $\{ y, z, \dot{x}, \dot{y}, \dot{z} \}$ fixed;

$$\partial \pounds / \partial x = - \partial U / \partial x = F_x(\mathbf{r}) \,.$$

Now, the other partial derivative :

$\partial / \partial \dot{x}$ means

vary $\dot{x}$ but keep the all other 5 variables $\{ x, y, z, \dot{y}, \dot{z} \}$ fixed;

$$\partial \pounds / \partial \dot{x} = \partial T / \partial \dot{x} = m \dot{x}$$

Newton's second law: $F_x(x) = m\ddot{x} \Rightarrow$

$$\boxed{\frac{\partial \pounds}{\partial x} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{x}}}$$

Similarly for *y* and *z* ;

$$\frac{\partial \pounds}{\partial y} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{y}} \;\color{red}{;\ \&}\; \frac{\partial \pounds}{\partial z} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{z}}$$

These are Lagrange's equations..

Lagrange's equations

For unconstrained motion,

$$\frac{\partial £}{\partial \mathbf{r}} = \frac{d}{dt}\,\frac{\partial £}{\partial \dot{\mathbf{r}}} \qquad \text{(3 eqs.)}$$

for $\mathbf{r} = \{x, y, z\}$,

where $£ = T - U$.

Remember the meanings of the partial derivatives!

$\partial/\partial x$ means vary x but keep the other 5 variables fixed;

$\partial/\partial \dot{x}$ means vary $\dot{x}$ but keep the other 5 variables fixed.

You should see that the equation looks like the Euler -Lagrange equation. Then what is the variational problem?

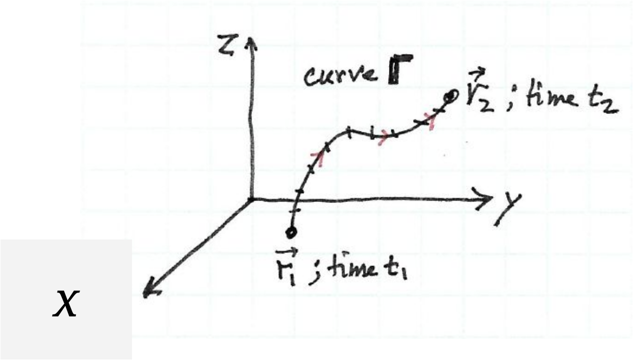
Hamilton's action integral

Define the action integral S by

$$S(\Gamma) = \int_{t_1}^{t_2} £(\mathbf{r}, \dot{\mathbf{r}})\, dt$$

where:

- Γ is a path in space from $\mathbf{r}_1$ to $\mathbf{r}_2$
- $\mathbf{r}(t)$ is a function of time that traverses the path as $t : t_1 \rightarrow t_2$
- Important: $\mathbf{r}(t_1) = \mathbf{r}_1$ and $\mathbf{r}(t_2) = \mathbf{r}_2$.

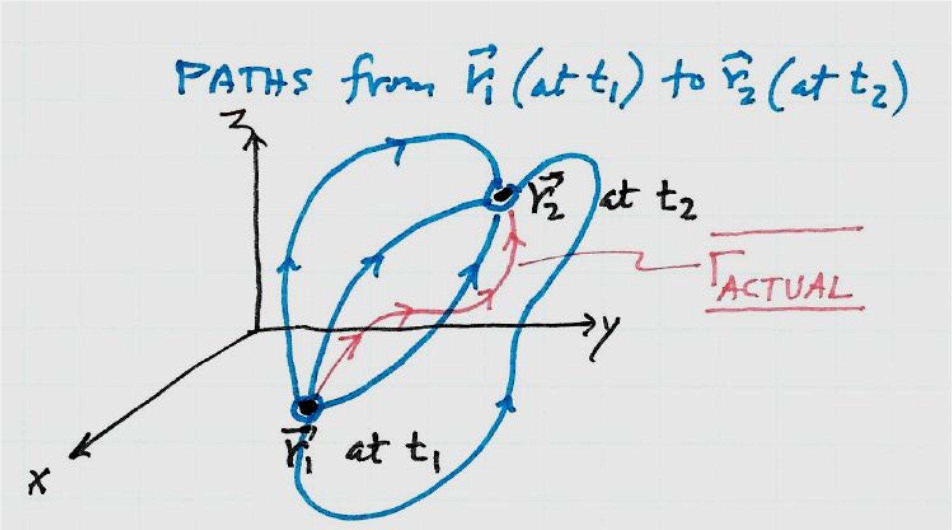


3

Hamilton's Principle

The actual path taken by *m* under the influence of the force $-\nabla U$, in order to move from $(t_1 , \mathbf{r_1})$ to $(t_2 , \mathbf{r_2})$, will be the path Γ_{actual} for which S is minimum.

"least action"



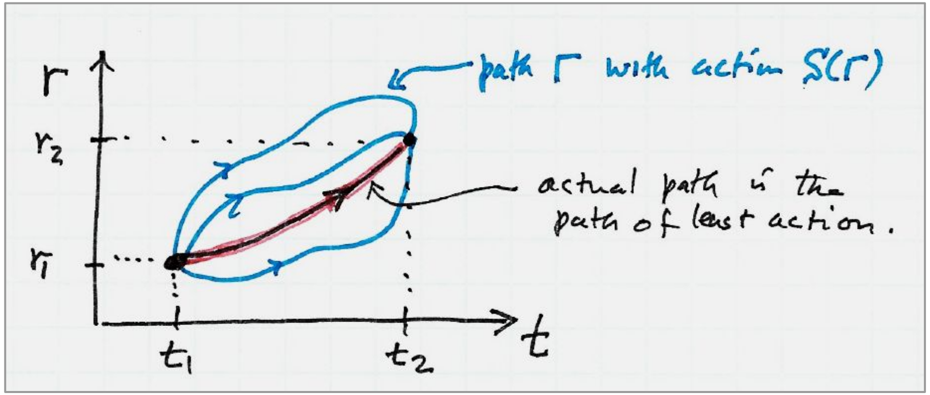
Hamilton's Principle

Suppose the particle moves from (t_1 , $\mathbf{r_1}$) to (t_2 , $\mathbf{r_2}$), under the influence of the force $\mathbf{F} = -\nabla U$. The trajectory of the particle is $\mathbf{r}(t)$, which defines a path Γ_{actual} .

Hamilton's Principle states

$$\min_{\{\Gamma\}} S(\Gamma) = S(\Gamma_{actual})$$

Of all the paths from (t_1 , $\mathbf{r_1}$) to (t_2 , $\mathbf{r_2}$), the particle follows the ***path of least action.***



Note: The endpoints are fixed in both space and time.

Proof of Hamilton's Principle

$$S(\Gamma) = \int_{t_1}^{t_2} \pounds(\mathbf{r}, \dot{\mathbf{r}})\, dt$$

What do I need to prove?

min $S(\Gamma)$ occurs when $\mathbf{r}(t)$ obeys Lagrange's equations __

$$\frac{\partial \pounds}{\partial \mathbf{r}} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{\mathbf{r}}}$$

- The minimum over all paths Γ [from $(t_1, \mathbf{r_1})$ to $(t_2, \mathbf{r_2})$] has $\delta S = 0$.
- ***The calculus of variations***;

$$\delta S = \int_{t1}^{t2} \{ (\partial \pounds / \partial \mathbf{r}) \cdot \delta \mathbf{r} + (\partial \pounds / \partial \dot{\mathbf{r}}) \cdot \delta \dot{\mathbf{r}} \}\, dt$$

$$\text{2nd term} = d/dt\, [(\partial \pounds / \partial \dot{\mathbf{r}}) \cdot \delta \mathbf{r}] - d/dt\, [(\partial \pounds / \partial \dot{\mathbf{r}})] \cdot \delta \mathbf{r}$$

We require $\delta \mathbf{r} = 0$ at the endpoints, so the integral of d/dt [...] is zero.

$$\therefore\ \delta S = \int_{t1}^{t2} \{ (\partial \pounds / \partial \mathbf{r}) - d/dt\, (\partial \pounds / \partial \dot{\mathbf{r}}) \} \cdot \delta \mathbf{r}\, dt$$

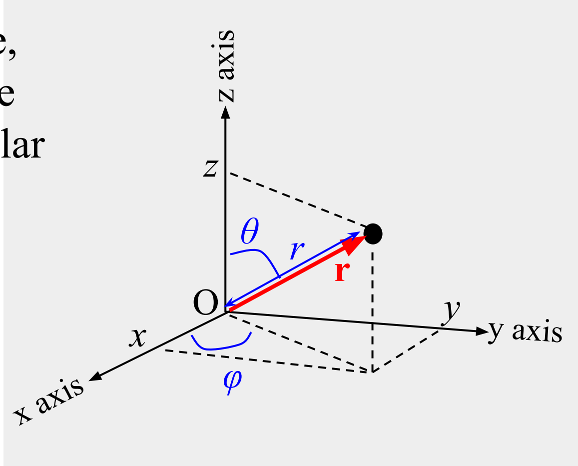
- δS must be = 0 for any variation of the path, i.e., for any function $\delta \mathbf{r}(t)$. The only way that can be true is if the function in {..} brackets is 0.
- For the least action, $\mathbf{r}(t)$ obeys Lagrange's equation.

4 Generalized coordinates

We can always use Cartesian coordinates {x, y, z} to specify the trajectory of the particle.

But suppose some other coordinates could be used, say, $\{ q_1 , q_2 , q_3 \}$.

For example, we could use spherical polar coordinates $\{ r, \theta, \varphi \}$.



We would have a 1-to-1 correspondence between $\{ q_1 , q_2 , q_3 \}$ and $\{ x , y , z \}$.
That is, ∃ functions

$$q_i = q_i(\mathbf{r}) \qquad \text{for} \quad i = 1, 2, 3$$

or

$$\mathbf{r} = \mathbf{r}(q_1 , q_2 , q_3) .$$

Then we could write

$$£ = £(q_1,q_2,q_3 , \dot{q}_1,\dot{q}_2,\dot{q}_3)$$

and

$$S = \int_{t_1}^{t_2} £(q_1\ q_2\ q_3\ \dot{q}_1\ \dot{q}_2\ \dot{q}_3)\, dt$$

The actual path of the particle has least action, $\delta S = 0$; that's Hamilton's principle.

The equation $\delta S = 0$ gives us Lagrange's equations, but now in terms of $\{ q_1 , q_2 , q_3 \}$.

So the equations of motion in terms of any set of generalized coordinates, are

$$\frac{\partial \pounds}{\partial q_i} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{q}_i}$$

3 equations; $i = 1\ 2\ 3$

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\pounds = T - U$
4. Write down Lagrange's equations.
5. Solve the equations.

Example 7.2 from Taylor

using Plane Polar Coordinates

FIGURE 7.1

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

$z = 0$

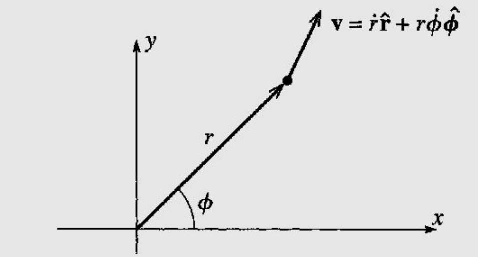


Figure 7.1 The velocity of a particle expressed in two-dimensional polar coordinates.

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U_c(x, y)$$
$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$$

<u>The r equation</u>

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$$

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = m\ddot{r} = F_r + mr\dot{\phi}^2$$

$$a_r = \ddot{r} - r\dot{\phi}^2$$

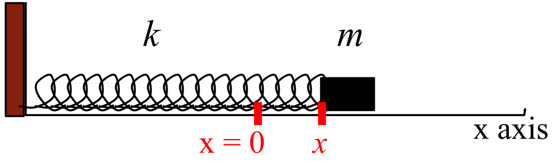
<u>The ϕ equation</u>

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}\underbrace{\left(mr^2\dot{\phi}\right)}_{\text{ang. momentum } L_z} = \underbrace{rF_\phi}_{\text{torque } N_z}$$

Problem 7.2 from Taylor

"Write down the Lagrangian for a one-dimensional particle moving along the x axis and subject to a force $F = -k\,x$ (with k positive). Find the Lagrange equation of motion and solve it."



$$\pounds = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2$$

$$\partial \pounds / \partial x = (d/dt)\ \partial \pounds / \partial \dot{x}$$

$$-k x = (d/dt)\, m \dot{x} = m \ddot{x}$$

$$\ddot{x} = -\omega^2 x \Rightarrow x(t) = A \cos(\omega t - \delta)$$

Homework Assignment 12
due in class Wednesday November 22

[61] Problem 7.2 *
[62] Problem 7.3 *
[63] Problem 7.8 **
[64] Problem 7.14 *
[65] Problem 7.21 *
[66] Problem 7.31 **
[67] Problem 7.43 *** [computer]

Use the cover sheet.