Chapter 7. Lagrange's Equations

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. £ = T − U
4. Write down Lagrange's equations.
5. Solve the equations.

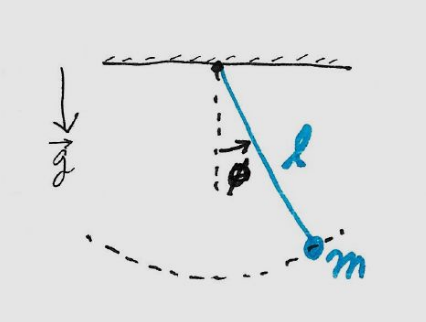
Lagrange's equations are

$$\frac{\partial £}{\partial q_i} = \frac{d}{dt}\frac{\partial £}{\partial \dot{q}_i} \qquad \{ i = 1\ 2\ 3\ \ldots \}$$

Section 7.2. Constrained Systems; an Example

"Constrained motion" means that the particle is not free to move throughout the space; its motion is limited by certain constraints.

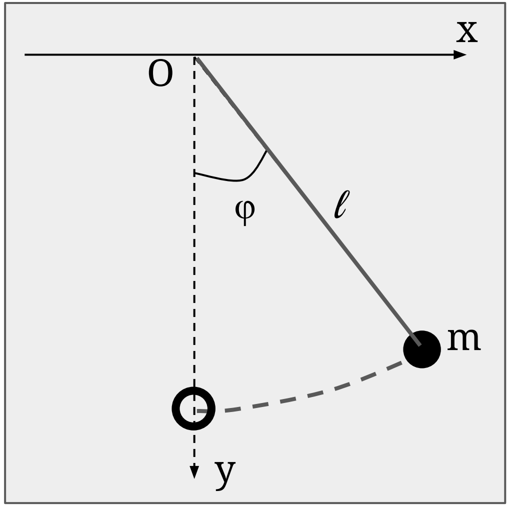
For example, consider the pendulum…



The length of the rod (or string) is constant (= l) so the mass m can only move on a circle or arc of radius l .

Hamilton's Principle ("least action") still applies; ⇒ Lagrange's equations.

Example: The Plane Pendulum



$x = l \sin\varphi$
$y = l \cos\varphi$

$mgh = mg(l - y)$

The generalized coordinate is φ.

$\pounds = T - U$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m l^2 \dot{\varphi}^2$

$U = m g (l - y) = m g l (1 - \cos\varphi)$

$$\pounds = \frac{1}{2} m l^2 \dot{\varphi}^2 - m g l (1 - \cos\varphi)$$

Lagrange's equation, in terms of the generalized coordinate, φ …

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$-mgl \sin\phi = \frac{d}{dt}\left(ml^2 \dot{\phi}\right) = ml^2 \ddot{\phi}$$

$$\ddot{\phi} = -\frac{g}{\ell} \sin\phi$$

We are familiar with this equation from earlier calculations.

The solution is an "elliptic integral" ; Taylor Problem 4.28.

Section 7.3: Constrained Systems in General

To be general, consider a system of N particles:

※ labels $\alpha = \{1\ 2\ 3 \ldots N\}$

※ positions $\mathbf{r}_\alpha = \{\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\ \ldots\ \mathbf{r}_N\}$

※ generalized coordinates

$$q_i = \{q_1\ q_2\ q_3\ \ldots\ q_n\} \quad (i = 1 \ldots n)$$

The number of particle coordinates is 3N (for a three dimensional system). The number of generalized coordinates is smaller, call it n, because of constraints.

※ ∃ necessary functional relationships

$\mathbf{r}_\alpha = \mathbf{r}_\alpha(q_1, \ldots, q_n; t)$ for $\alpha = \{1\ 2\ 3 \ldots N\}$

$q_i = q_i(\mathbf{r}_1, \ldots, \mathbf{r}_N; t)$ for $i = \{1\ 2\ 3 \ldots n\}$

Note the possible (but not always necessary) time-dependence of the relations.

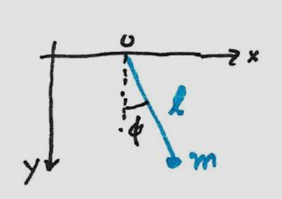
[[Side comment: Taylor won't use the terms *scleronomous* coordinates and *rheonomous* coordinates; instead he calls them "natural" and "nonnatural". - - Footnote 4 on page 249.]]

Taylor gives some examples:

- the plane pendulum

  x, y ; $N = 2$

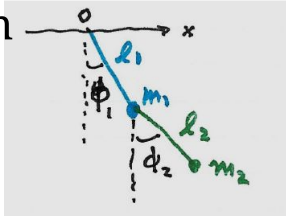
  φ ; $n=1$



- the double plane pendulum

  x_1, y_1, x_2, y_2 ; $N = 4$

  φ_1, φ_2 ; $n=2$

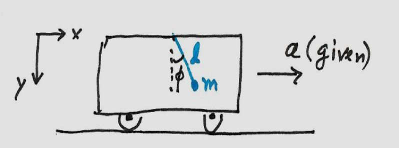


- a pendulum in a railroad car with specified acceleration a

  x, y ; $N = 2$

  φ ; $n=1$

  with *time dependent* relations



"Degrees of Freedom"

- ❑ n is the number of degrees of freedom, i.e., the number of coordinates that can vary independently.
- ❑ N = the number of mass points.
- ❑ 3N = the number of Cartesian coordinates
- ❑ $n \leq 3N$

For a ***rigid body***, n = 6 while N = infinite.

"Holonomic systems" : n is the number of degrees of freedom ***and*** n is the number of generalized coordinates.

"Nonholonomic systems" (Taylor gives a rolling ball as an example) will not be considered in this course.

Section 7.4.
Prove Lagrange's Equations with Constraints

To make it simple, consider a particle that is constrained to move on a surface.

There are two generalized coordinates, q_1 and q_2.

The net constraining force is $\mathbf{F}_{cstr}$.
All other forces can be derived from a potential energy function, which may depend on time. So the force on the particle is

$$\mathbf{F}_{total} = \mathbf{F} + \mathbf{F}_{cstr} \quad \text{and} \quad \mathbf{F} = -\nabla U(\mathbf{r},t) \ .$$

Let £ = T − U.

The action integral

- Let **r**(t) = the actual path followed by the particle under the influence of the forces.

- Let **R**(t) = **r**(t) + **ε**(t) where **ε**(t) describes a small variation of the path; i.e., infinitesimal; *and* **R** obeys the constraints.

- The action integral for **R**(t) is

$$S = \int_{t1}^{t2} \pounds\,(\mathbf{R}, \dot{\mathbf{R}}, t)\, dt\ ;$$

and $S_0 = \int_{t1}^{t2} \pounds\,(\mathbf{r}, \dot{\mathbf{r}}, t)\, dt$ = the minimum.

Now $\delta S = S - S_0 = \int_{t1}^{t2} \delta \pounds\ dt$

- $\delta \pounds = \pounds\,(\mathbf{R}, \dot{\mathbf{R}}, t) - \pounds\,(\mathbf{r}, \dot{\mathbf{r}}, t)$

$= \frac{1}{2} m \left[(\dot{\mathbf{r}} + \dot{\boldsymbol{\varepsilon}})^2 - \dot{\mathbf{r}}^2 \right] - [U(\mathbf{r} + \boldsymbol{\varepsilon}) - U(\mathbf{r})]$

$= m\, \dot{\mathbf{r}} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \cdot \nabla U \quad + \quad O\,(\boldsymbol{\varepsilon}^2)$

$= m\, d/dt\, (\dot{\mathbf{r}} \cdot \boldsymbol{\varepsilon}) - m\, \ddot{\mathbf{r}} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot \mathbf{F}$

integrates to 0 because $\boldsymbol{\varepsilon}(t_1) = 0$ and $\boldsymbol{\varepsilon}(t_2) = 0$.

$=$ drop $- \boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}}$

- $$\delta S = - \int_{t1}^{t2} \boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}}\, dt$$

- *The constraint force is normal to the surface*; therefore

$$\boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}} = (\mathbf{R} - \mathbf{r}) \cdot \mathbf{F}_{\mathbf{cstr}} = \mathbf{0}.$$

- *Thus the action integral is stationary , δS = 0 , at the actual path of the particle,* ***r****(t).*

Theorem.
The generalized coordinates obey Lagrange's equations.

Proof.
We just proved that Hamilton's principle ($\delta S = 0$) holds for all variations of the path ***that obey the constraints.***

Any variation of the generalized coordinates, q_1 and q_2 , obeys the constraints.

Write S in terms of the generalized coordinates,

$$S = \int_{t1}^{t2} \pounds (q_1\ q_2\ \dot{q}_1\ \dot{q}_2\ ; t)\, dt\ .$$

Then we have $\delta S = 0$ for any variations of $q_1(t)$ and $q_2(t)$.

By the calculus of variations (Chapter 6) $q_1(t)$ and $q_2(t)$ must obey the Euler-Lagrange equations; i.e.,

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) \quad \text{AND} \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right)$$

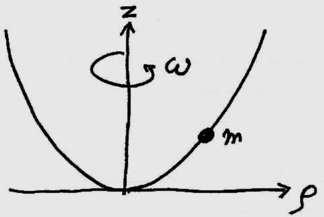
For a holonomic system,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad \text{for} \quad i = 1\ 2\ 3 \ldots\ n$$

where $\pounds = T - U$.

Example: Problem 7.41

BEAD ON A STIFF SPINNING WIRE



BEAD ON A FRICTIONLESS WIRE

Cylindrical coordinates of m (ρ, ϕ, z)

Constraints on m

$$z = k\rho^2$$

$$\phi = \omega t$$

■ The Lagrangian $\pounds(\rho, \dot{\rho}; t)$

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{where} \quad x = \rho\cos\phi = \rho\cos\omega t, \quad Z = \rho\sin\phi = \rho\sin\omega t$$

$$T = \tfrac{1}{2}m\left[\dot{\rho}^2 + \rho^2\omega^2 + (2k\rho\dot{\rho})^2\right]$$

$$U = mgz = mgk\rho^2$$

$$\mathcal{L} = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right) - mgk\rho^2$$

"the centrifugal potential"

■ The equation of motion

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right)$$

$$(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 - (\omega^2 - 2gk)\rho = 0$$

■ The equilibrium positions

* $\dot{\rho} = 0$ and $\ddot{\rho} = 0$
* $\rho = 0$ is an equilibrium point
* Also, if $\omega^2 = 2gk$ then <u>any</u> ρ is an equilibrium point.

■ Stability analyses ; consider $\rho' = 0$ (EQ.) ;

Then $-\pounds = (mgk - \tfrac{1}{2}m\omega^2)\,\rho^2$;

- $\rho = 0$ is STABLE if (*&only if*) $\omega^2 < 2gk$
- Any $\rho > 0$ is a STABLE EQ. if $\omega^2 = 2gk$.

The trajectory of a particle moving in a potential obeys Lagrange's equations. For any set of generalized coordinates,

$$\frac{\partial \pounds}{\partial q_i} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{q}_i}$$

n equations; i = 1 2 3 ... n

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\pounds = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Homework Assignment 12
due in class Wednesday November 22

[61] Problem 7.2 *
[62] Problem 7.3 *
[63] Problem 7.8 **
[64] Problem 7.14 *
[65] Problem 7.21 *
[66] Problem 7.31 **
[67] Problem 7.43 *** [computer]

USE THE COVER SHEET.