HOMEWORK AND EXAMS

- *Homework Set 13*
  due Wednesday November 27

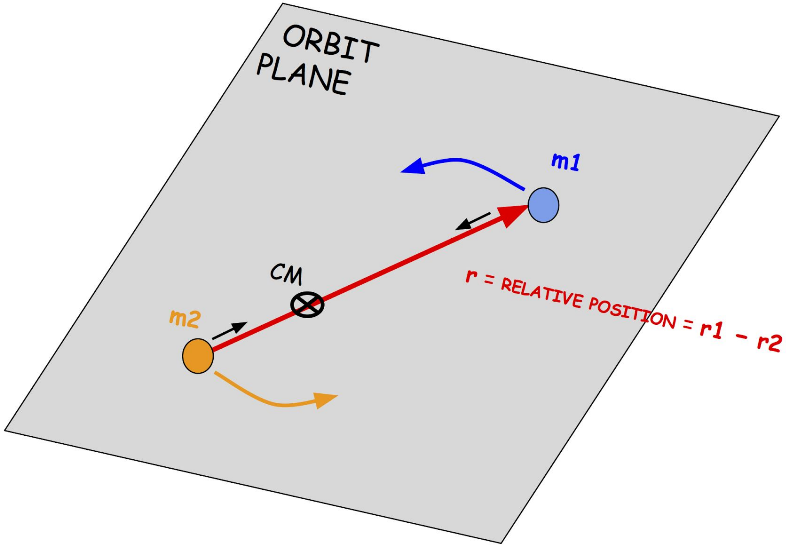
- *Exam 3*
  Monday December 4

- *Homework Set 14*
  due Friday December 8

- *Final Exam*
  Tuesday December 12

CHAPTER 8

TWO BODIES WITH A CENTRAL FORCE



ANGULAR MOMENTUM IS CONSERVED, SO THE 2 BODIES MOVE ON A PLANE, CALLED THE *ORBIT PLANE.*

THE CENTER OF MASS POSITION VECTOR AND THE RELATIVE POSITION VECTOR

$$\mathbf{R} = (m_1 \, \mathbf{r_1} + m_2 \, \mathbf{r_2}) / M$$

$$\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$$

$$\mathbf{r_1} = \mathbf{R} + (m_2/M) \, \mathbf{r}$$

$$\mathbf{r_2} = \mathbf{R} - (m_1/M) \, \mathbf{r}$$

8.5 The Equation of the Orbit

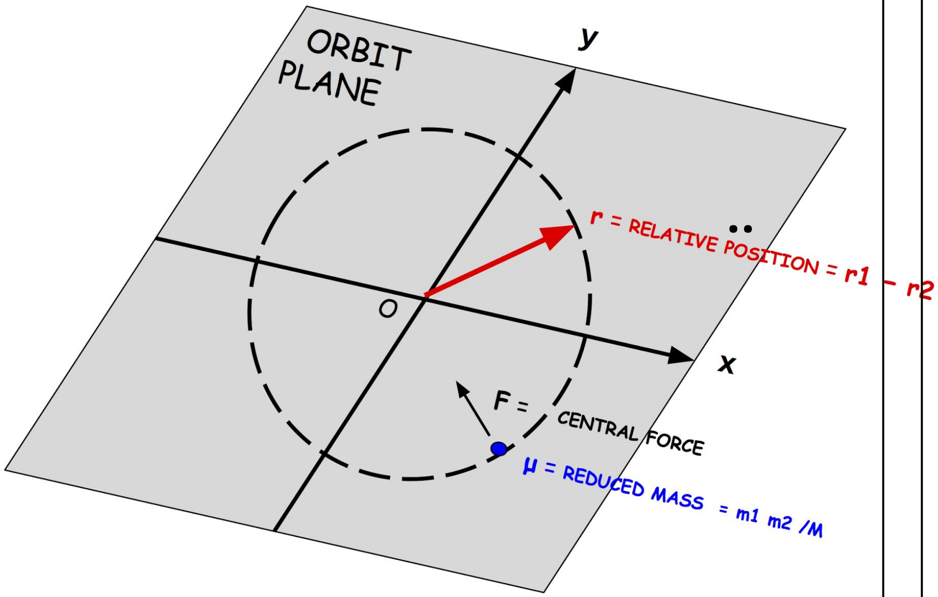
Review Sections 8.1 to 8.4

※ The 2-body problem reduces to 2 1-body problems:

※ The center of mass moves with constant velocity;

$$\mathbf{R}(t) = \mathbf{V}_c\, t\ .$$

※ The dynamics of the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, is $\mu\, \mathbf{r}'' = \mathbf{F}$, which is equivalent to a (fictitious) 1-body problem with mass $\mu = m_1\, m_2 / M$.



Plane polar coordinates, *r and φ*

2. In Lagrangian terms,

$$£_{rel} = \tfrac{1}{2}\, \mu(\dot{r}^2 + r^2\, \dot{\varphi}^2) - U(r)$$

3. There are two constants of the motion, ℓ and E:

$$\ell = \mu\, r^2\, \dot{\varphi}$$

$$E = \tfrac{1}{2}\, \mu\, (\dot{r}^2 + r^2\, \dot{\varphi}^2) + U(r)$$

$$E = \tfrac{1}{2}\, \mu\, \dot{r}^2 + \ell^2/(2\mu r^2) + U(r)$$

The equation for the orbit

- The "orbit" means the "trajectory"; i.e., the curve in space that the object moves on.
- The "orbit" means the curve, without time information.
- Today we'll try to calculate the orbit, and then later we'll add the time information.

- The equation for the orbit is a function $r(\varphi)$.

My method (different from Taylor)

Start with $\ell = \mu\, r^2\, \dot{\varphi}$ (1)

and

$$E = \tfrac{1}{2}\, \mu\, \dot{r}^2 + \ell^2/(2\mu r^2) + U(r). \qquad (2)$$

Now eliminate t like this:

$$\ell = \mu\, r^2\, (d\varphi/dt)$$

$$dt = \mu\, r^2\, (d\varphi) / \ell$$

so $\dot{r} = dr/dt = \dfrac{\ell\ \ dr}{\mu\, r^2\ \ d\varphi}$

Put that into (2).

My method (different from Taylor)

Start with $\ell = \mu\, r^2\, \dot{\varphi}$ (1)

and

$$E = \tfrac{1}{2}\mu\, \dot{r}^2 + \ell^2/(2\mu r^2) + U(r). \qquad (2)$$

Now eliminate t like this:

$$\ell = \mu\, r^2\, (d\varphi/dt)$$

$$dt = \mu\, r^2\, (d\varphi) / \ell$$

so $\dot{r} = dr/dt = \dfrac{dr}{(\mu\, r^2 / \ell)\ d\varphi}$

Put that into (2).

But first, let u = 1/r.

Then we'll calculate u(φ).
That will solve the problem,
because then r(φ) = 1 /u(φ) .

OK, $r = 1/u \;\Rightarrow\; dr = -(1/u^2)\, du$.

$$\frac{dr}{d\varphi} = -\frac{1}{u^2}\frac{du}{d\varphi}$$

$$\dot{r} = -\frac{\ell}{\mu r^2}\frac{1}{u^2}\frac{du}{d\varphi}$$

▪ *Also, let $\mu = m$ because μ looks too much like u.* ▪ *Note: $r^2 u^2 = 1$.*

$$E = \frac{1}{2}\mu \dot{r}^2 + \ell^2/(2\mu r^2) + U(r)$$

$$\dot{r} = -(\ell/m)\, du/d\varphi$$

$$E = \frac{1}{2} m (\ell/m)^2 (du/d\varphi)^2 + \ell^2 u^2/(2m) + U(r)$$

{ Don't confuse u and U;
$u = u(t) = 1/r(t)$
$U = U(r)$ the potential energy }

$$(du/d\varphi)^2 + u^2 = (2m/\ell^2)\,[\,E - U(r)\,]$$

8.6 Keplerian Orbits

★ U(r) is the gravitational potential energy.

$$U(r) = -G\, m_1\, m_2 /r$$

$$U(r) = -G\, M\, m\, u$$

Note: $m_1\, m_2 = M\, \mu$

$U(r) = -K\, u$ where $K = GMm$

- OK, here is the equation we had:

$$(du/d\varphi)^2 + (u - mK/\ell^2)^2$$

$$= 2mE/\ell^2 + (mK/\ell^2)^2 \quad \equiv \mathbf{A^2}$$

- and here is the solution:

$$u(\varphi) = mK/\ell^2 + \xi(\varphi)$$

$$= mK/\ell^2 + A\cos(\varphi - \varphi_0).$$

Thus the orbit equation in plane polar coordinates is

$$\frac{1}{r(\varphi)} = mK/\ell^2 + A\cos(\varphi - \varphi_0).$$

What curve is this?

I've introduced a bunch of constants.

Let's write them down again.

- ❑ $K = G\,M\,m = G\,m_1\,m_2$; this is not a variable .
- ❑ $A = \text{SQRT}[\,2mE/\ell^2 + (mK/\ell^2)^2\,]$; thus A is determined by E and ℓ .
- ❑ φ_0 = a boundary condition for the angle φ .
- ❑ ℓ = angular momentum
- ❑ E = energy; recall from last time that $E \leq 0$ for a bounded orbit.

Thus the orbit depends on three parameters … $\{\ell, E, \varphi_0\}$.

$$(du/d\varphi)^2 + u^2 = (2m/\ell^2)\,[\,E - U(r)\,]$$

$U(r) = -K\,u$ where $K = GMm$

$$(du/d\varphi)^2 + u^2$$
$$= (2m/\ell^2)\,[\,E - U(r)\,]$$
$$= (2mE/\ell^2) + (2mK/\ell^2)\,u$$

$$(du/d\varphi)^2 + (u - mK/\ell^2)^2$$
$$= 2mE/\ell^2 + (mK/\ell^2)^2 \equiv A^2$$

$$(du/d\varphi)^2 + (u - mK/\ell^2)^2 = A^2$$

Solution

Let $\xi(\varphi) = u(\varphi) - mK/\ell^2$.

Then

$$(d\xi/d\varphi)^2 + \xi^2 = A^2 .$$

Differentiate w.r.t. φ

$$2\,\xi'(\varphi)\,\xi''(\varphi) + 2\,\xi(\varphi)\,\xi'(\varphi) = 0.$$
$$\xi''(\varphi) = -\,\xi(\varphi)\,.$$

∴ We can write $\xi(\varphi) = C\cos(\varphi - \varphi_0)$.

Now… $\xi'(\varphi)^2 + \xi(\varphi)^2$

$$= C^2\sin^2 + C^2\cos^2 = C^2 = A^2$$

C = A

The orbit equation in plane polar coordinates is

$$\frac{1}{r(\varphi)} = mK/\ell^2 + A\cos(\varphi - \varphi_0).$$

What curve is this?

Recall,

$$A^2 = 2mE/\ell^2 + (mK/\ell^2)^2 .$$

E is < 0, so $A < (mK/\ell^2)$.

Let's write $A = \varepsilon\,(mK/\ell^2)$ where $\varepsilon < 1$;

i.e., we define another constant ε,

and use that instead of A.

So now we have

$$\frac{1}{r(\varphi)} = (mK/\ell^2)\,[\,1 + \varepsilon\cos(\varphi - \varphi_0)\,]$$

Analyze the curve:

$$-1 \le \cos \le 1 \text{ and } \quad \varepsilon < 1\,;$$

therefore

$$r_{min} \le r \le r_{max}$$

where

$$r_{min} = \frac{\ell^2}{mK(1+\varepsilon)} \quad \textit{(perihelion)}$$

$$r_{max} = \frac{\ell^2}{mK(1-\varepsilon)} \quad \textit{(aphelion)}$$

QUIZ QUESTION

The orbit equation in plane polar coordinates is

$$1/r = mK/\ell^2 \{ 1 - \varepsilon \cos(\varphi - \varphi_0)\}.$$

Sketch a graph of the orbit,
and label some points on the curve.

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