HOMEWORK AND EXAMS

- *Homework Set 13*
  due Wednesday November 29

- *Exam 3*
  Monday December 4

- *Homework Set 14*
  due Friday December 8

- *Final Exam*
  Tuesday December 12

Section 8.6.

The Kepler orbits

Read Section 8.6.

We need these results from Section 8.5

$$\ell = \mu r^2 \dot{\phi} \quad \text{and} \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$r = \frac{1}{u} \quad \text{and} \quad \mu = m$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{2m}{\ell^2}\left[E - U(r)\right]$$

Keplerian orbits

From Newton's theory of gravitation,

$$F_r(r) = -G\, m_1\, m_2 / r^2 \;\; -K/r^2$$

The potential energy is

$$U(r) = -\;K/r\,,$$

where $K = Gm_1\, m_2 \; = GM\mu$

The orbit equation, in terms of u (= 1/r)

is

$$\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{mK}{\ell^2}\right)^2 = \frac{2mE}{\ell^2} + \left(\frac{mK}{\ell^2}\right)^2 \equiv A^2$$

$$\frac{1}{r} = u = \frac{mK}{\ell^2} + A\cos(\varphi - \varphi_0)$$

$$u''(\varphi) = -\,u(\varphi) + mK / l^{\,2}$$

inhomogeneous linear diff. eq.

The solution is

particular + homogeneous sol.

$$u(\varphi) = mK / l^2 + A \cos(\varphi - \varphi_0)$$

where $A^2 = 2mE / l^2 + (mK / l^2)^2$

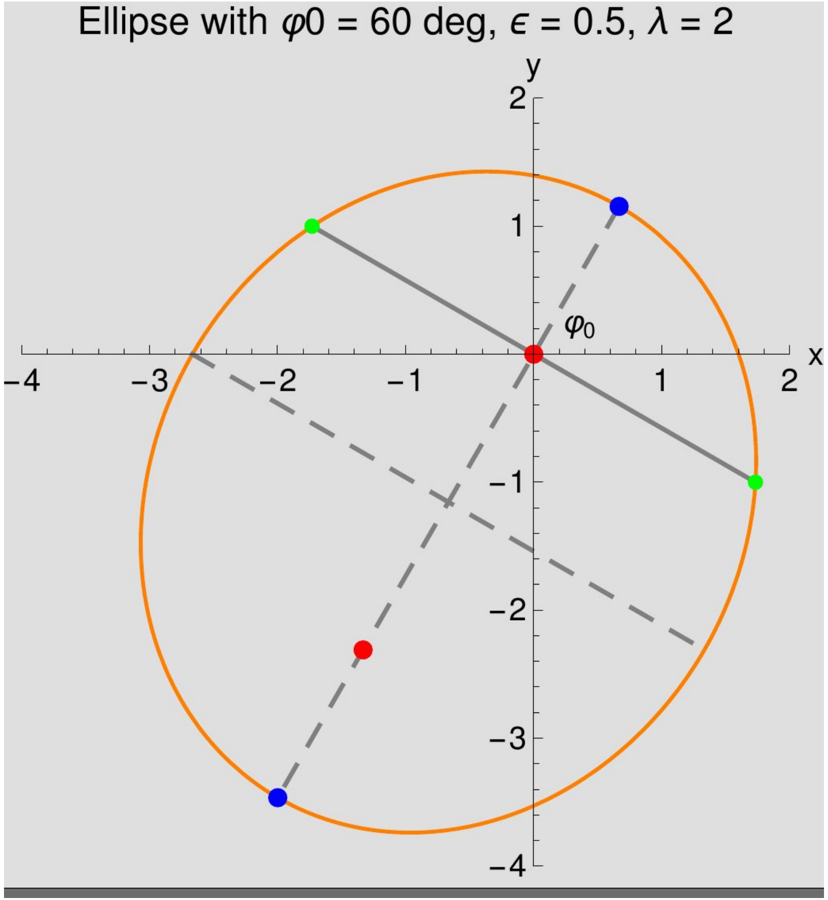
and $A = \varepsilon\, mK / l^2$ with $\varepsilon < 1$.

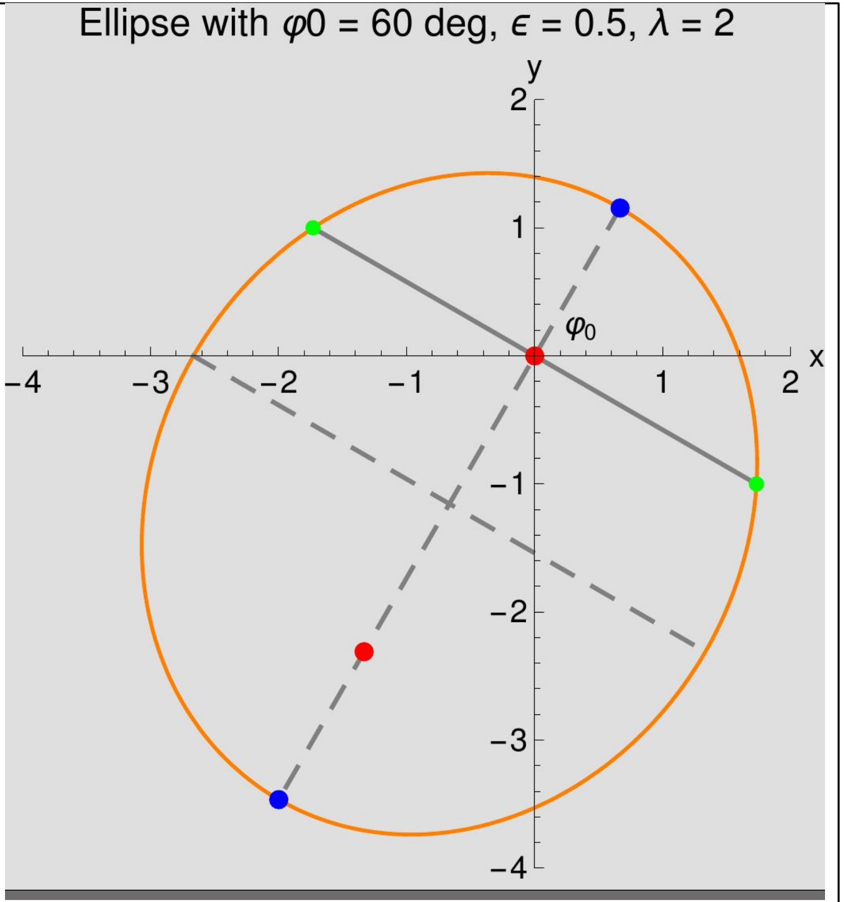
Ellipse geometry

Theorem. The polar equation

$$1/r = (1/\lambda)\ \{ 1 + \varepsilon \cos(\varphi - \varphi_0) \}$$

is an ellipse with orientation φ_0, eccentricity ε, and semi-latus rectum λ.





$r(\varphi) = \lambda \ / \ [\, 1 + \varepsilon \cos(\varphi - \varphi_0) \ ]$

<u>Ellipse terminology</u>

- Blue points = perihelion and aphelion
- Red points = the focal points
- Long dashed line = major axis;
  2a = length of the major axis;
  a = semi-major axis
- **eccentricity** ε = D(f1,f2) / 2a
- Short dashed line = minor axis;
  2b = length of the minor axis;
  b = the semi-minor axis
- **latus rectum** = a chord that passes through the focus of a conic and is perpendicular to the major axis;
  = the line segment connecting the green points ; its length is 2λ;
  λ = the semi-latus rectum
- Orientation angle = φ_0

Theorem. The polar equation

$$1/r = (1/\lambda)\ \{ 1 + \varepsilon \cos(\varphi - \varphi_0) \}$$

is an ellipse with orientation φ_0, eccentricity ε, and semi-latus rectum λ.

Proof.

- W.L.O.G. let $\varphi_0 = 0$.
- $r = \lambda / \{ 1 + \varepsilon \cos \varphi \}$
- It's a closed curve;

  as $\varphi : 0 \to \pi \to 2\pi$

  $r : r_{min} \to r_{max} \to r_{min}$
- $r_{min} = \lambda/(1+\varepsilon)$ and $r_{max} = \lambda/(1-\varepsilon)$
- In fact it's an ellipse. **(proof on next slide)**
- The origin = the center of attraction is a focal point of the ellipse.
- Eccentricity (ε) and semilatus rectum (λ)

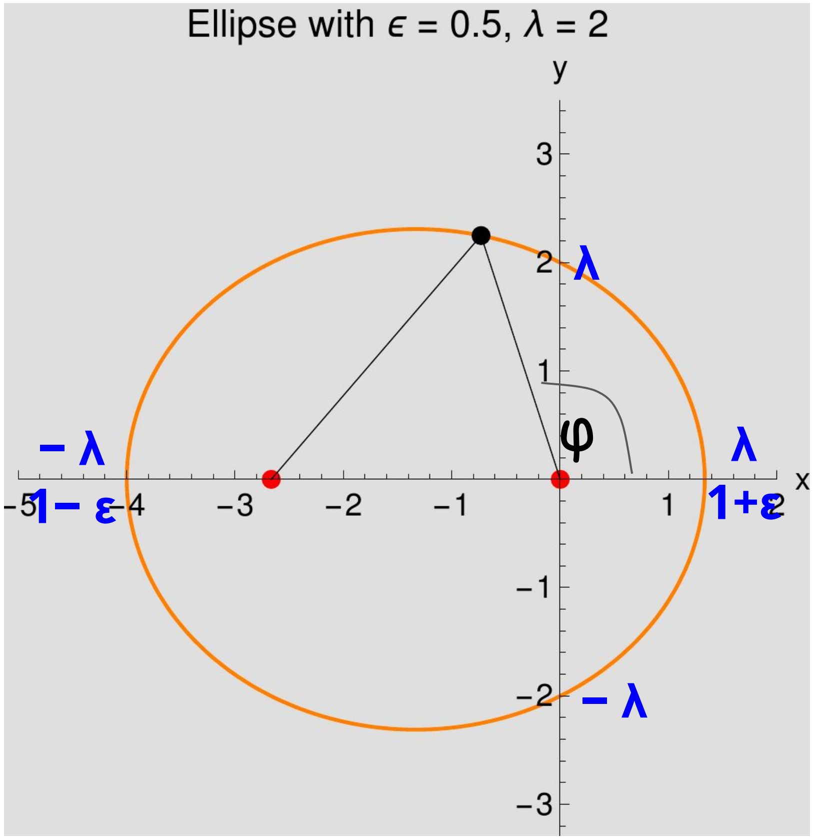
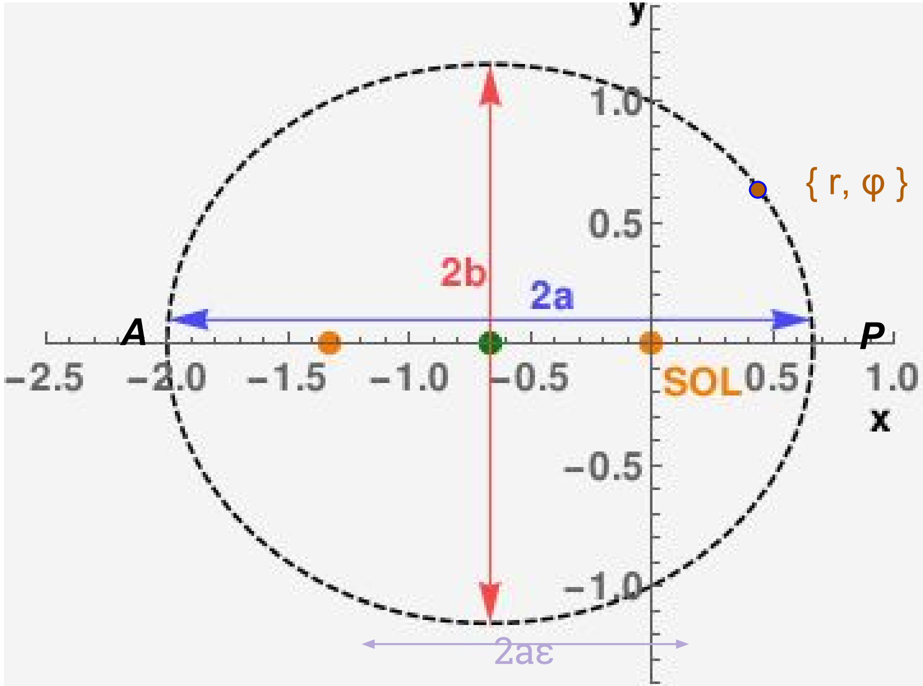


FIGURE 8.10
ELLIPSE GEOMETRY



Problem 8.16: Prove $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$

Given $r = \frac{\lambda}{1+\epsilon\cos\phi}$

$x = r\cos\phi$ and $y = r\sin\phi$.

Now relate y and x in the form of an ellipse $\left(\frac{x+d}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

$$\lambda = r + \epsilon x$$

$$r = \lambda - \epsilon x$$

$$x^2 + y^2 = \lambda^2 - 2\lambda\epsilon x + \epsilon^2 x^2$$

$$y^2 = -(1-\epsilon^2)x^2 - 2\lambda\epsilon x + \lambda^2$$

$$= -(1-\epsilon^2)\left[x + \frac{\lambda\epsilon}{1-\epsilon^2}\right]^2 + (1-\epsilon^2)\left(\frac{\lambda\epsilon}{1-\epsilon^2}\right)^2 + \lambda^2$$

$$= -(1-\epsilon^2)[x+d]^2 + \frac{\lambda^2}{1-\epsilon^2}$$

$$d = \frac{\lambda\epsilon}{1-\epsilon^2}$$

$$\left(\frac{y}{b}\right)^2 + \left(\frac{x+d}{a}\right)^2 = 1$$

$b = \frac{\lambda}{\sqrt{1-\epsilon^2}}$ and $a = \frac{\lambda}{1-\epsilon^2}$

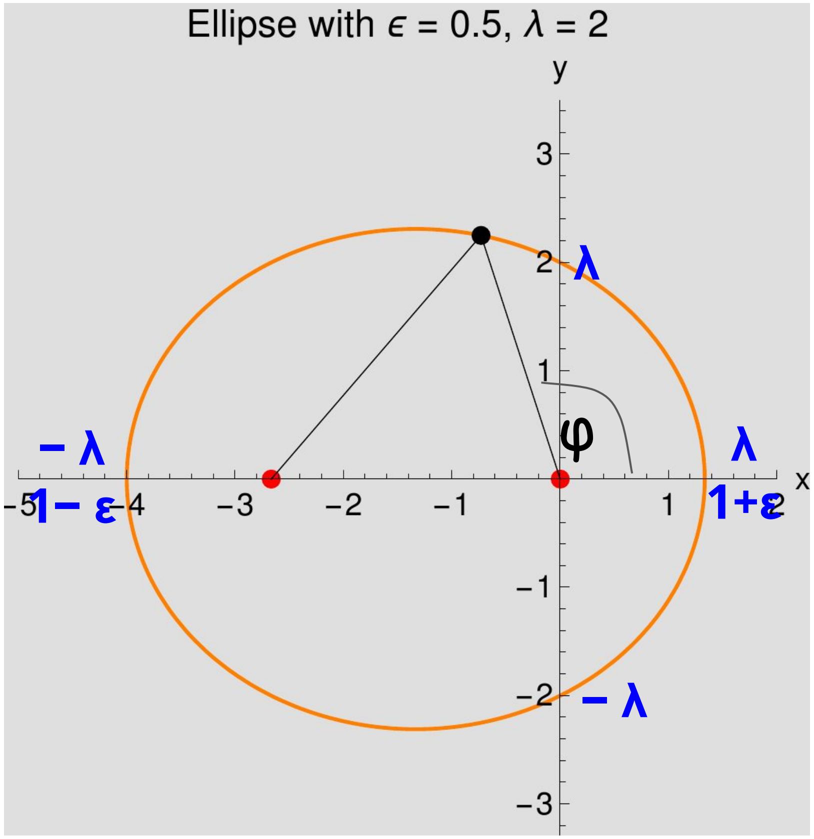
To make it simple, ***let*** $\varphi_0 = 0$.
I.e., set up the xy-coordinate system such that the perihelion is on the positive x axis.

The orbit equation in polar coordinates is

$$r(\varphi) = \frac{\lambda}{1 + \varepsilon \cos\varphi}$$

See FIGURE 8.9 : 1/r versus φ.

- Major axis : $2a = 2\lambda \;/\; (1-\varepsilon^2)$
- Distance between FP's :
  $D12 = 2a - 2\lambda \;/\; (1+\varepsilon) = 2\lambda\varepsilon \;/\; (1-\varepsilon^2)$
- Eccentricity = $D12 \;/\; (2a) = \varepsilon$



Example 8.4 : Halley's Comet

Edmund Halley (1656 – 1742) was a famous astronomer, and a friend of Isaac Newton.
Pronunciation of his name : rhymes with valley.

Interesting history.

One day (1687) Halley visited Newton, who was at that time an obscure math professor at Cambridge University. Halley asked Newton if he could explain Kepler's 3 laws of planetary motion. Newton said he figured it out a few years earlier. But he couldn't find the paper with his calculations, and he promised to send Halley a copy.

When Halley saw it, he was amazed that Newton had solved the most important problem in astronomy, but had never told anyone about it. So Halley arranged to get Newton's work published.

That is how Newton became famous.

Then Halley applied Newton's theory to observations of comets; and he proved that comets revolve around the sun and return to the sun after each period of revolution.

Observations of the comet

1531 (known to Halley)
~76 years
1607 (")
~75 years
1682 (" ; did Halley & Newton see it?)
~76 years
so he predicted it would return in 1758
most recent 1986 (I saw it in a telescope)
~75 years
next time 2061

Before Newton and Halley, people thought these were all different comets. But it's just the same one, seen near perihelion.

Wikipeida: ***Halley's Comet*** *or* ***Comet Halley****, officially designated* ***1P/Halley****, is a short-period comet visible from Earth every 75–76 years.*

Example 8.4 : Halley's Comet

$\varepsilon = 0.967$

perihelion = 0.59 AU

Compare:

Mercury has $0.31 < r < 0.47$ AU
Earth has $0.983 < r < 1.017$ AU
Neptune has $29.8 < r < 30.3$ AU

Calculate the aphelion distance for the orbit of Halley's comet.

$$r_{min} = \frac{C}{1+\varepsilon} \quad \text{and} \quad r_{max} = \frac{C}{1-\varepsilon}$$

$$r_{max} = \frac{1+\varepsilon}{1-\varepsilon}\, r_{min} = \frac{1.967}{0.033}\, 0.59\ \text{AU}$$

$$r_{max} = 35.2\ \text{AU}$$

The period of revolution of a planet, comet, satellite, etc.

This was discovered by Kepler.
By analyzing observations, he discovered ...

Kepler's third law (published 1619)
Kepler claimed that $\tau^2 \propto a^3$ for all the planets;

τ = period of revolution
a = semimajor axis

In other words, τ^2 / a^3 is the same for all the planets.

It's not precisely true, but it's close. Given the observations available to Kepler, it is remarkable that he came so close.

(next lecture)

Calculate the period of Halley's Comet.

By Kepler's 3rd law,

$$\frac{\tau_H^2}{a_H^3} = \frac{\tau_E^2}{a_E^3} = \frac{(1\ \text{year})^2}{(1\ \text{AU})^3}$$

$$a_H = \frac{1}{2}(r_{min} + r_{max}) = \frac{1}{2}(0.59 + 35.2)\,\text{AU}$$

$$= 17.9\ \text{AU}$$

$$\therefore \tau_H = 1\ \text{year}\left[\frac{17.9}{1}\right]^{3/2} = 75.7\ \text{year}$$

Homework Assignment 14
due in class Friday December 9

[76] Problem 8.19 ★★
[77] Problem 8.25 ★★★
[78] Problem 8.27 ★★★
[79] Problem 8.28 ★
[80] Problem 8.34 ★★

USE THE COVER SHEET.