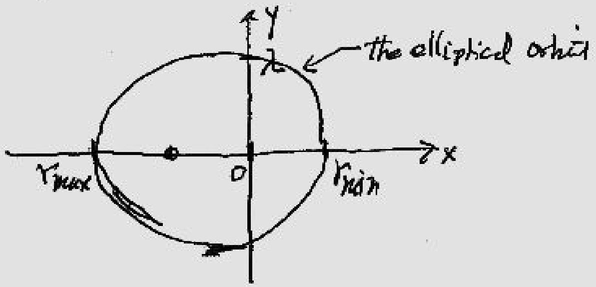
Section 8.6. *Bounded Kepler orbits*
Section 8.7. *Unbounded Kepler Orbits*
Read Sections 8.6 and 8.7.

- ***Review the equations so far***

• The orbit



$$r(\phi) = \frac{\lambda}{1+\epsilon\cos\phi}$$

$$r_{min} = \frac{\lambda}{1+\epsilon} \text{ and } r_{max} = \frac{\lambda}{1-\epsilon}$$

$$a = \text{semimajor axis} = \frac{1}{2}(r_{min}+r_{max})$$

$$a = \frac{\lambda}{1-\epsilon^2}$$

$$\text{Exercise} \quad \epsilon = \frac{r_{max}-r_{min}}{r_{max}+r_{min}}$$

- ***ℓ and E for Keplerian orbits***
  - Use $\{a,\varepsilon\}$ to define the ellipse.
  - Relate energy (E) and angular momentum (ℓ) to semimajor axis (a) and eccentricity (ε).

• angular momentum $\ell = \mu r^2\dot{\phi}$

Recall $\dfrac{1}{\lambda} = \dfrac{mK}{\ell^2}$ ← $m = \mu$

$\epsilon\dfrac{mK}{\ell^2}$

$$\left\{ u = \frac{1}{r} = \frac{mK}{\ell^2} + A\cos\phi \right\}$$

$$\therefore \ \ell^2 = mK\lambda = mKa(1-\varepsilon^2)$$

depends on both a and ε.

• energy $E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{K}{r}$

When $\phi = 0$, $\dot{r} = 0$ because $r = r_{min}$.

$$\therefore E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{K}{r_{min}}$$

$$E = \frac{m K a (1-\epsilon^2)}{2\mu \lambda^2/(1+\epsilon)^2} - \frac{K}{\lambda/(1+\epsilon)}$$

(remember: $m = \mu$).

$$E = \frac{K}{2}\frac{(1+\epsilon)^2}{\lambda} - \frac{K(1+\epsilon)}{\lambda} = \frac{K}{2\lambda}(1+\epsilon)(\epsilon-1)$$

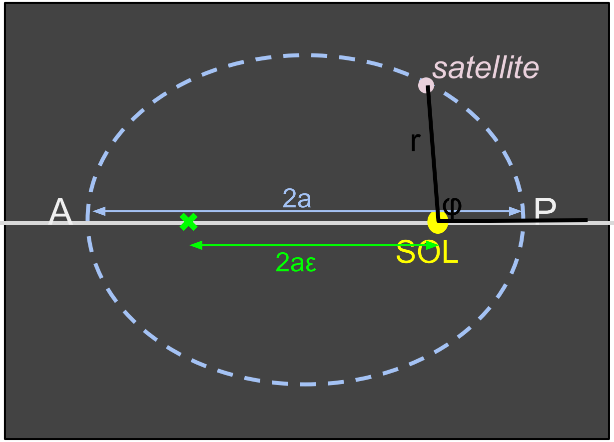
$$E = -\frac{K}{2a} \quad \text{(INDEPENDENT OF } \varepsilon\text{!)}$$

Results

$$\ell^2 = \mu K a (1-\varepsilon^2)$$

$$E = -K/(2a)$$

$$K = G m_1 m_2 = G M \mu$$



• ***Kepler's third law (1619)***

By analyzing Tycho's observations of the planets, Kepler concluded that $\tau^2 \propto a^3$ for all the planets; in other words, τ^2 / a^3 = **constant.**

It's not precisely true, but it is very close; recall Problem (8.15). (1) The derivation from Newton's theory, for circular orbits, is easy.

$$\mu \ddot{\vec{r}} = \frac{-K}{r^2}\hat{r}$$

$$-\frac{\mu v^2}{r} = \frac{-K}{r^2}$$

$$v = \sqrt{\frac{K}{\mu r}} = \sqrt{\frac{GM}{r}}$$

The speed is constant, so

$$\tau = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\tau^2 = \frac{4\pi^2 r^3}{GM}$$

Here $M = M_{sun} + M_{planet} \approx M_{sun}$;

so $\tau^2 \approx \dfrac{4\pi^2 r^3}{GM_{sun}}$ i.e. $\tau^2 \propto r^3$ APPROXIMATELY!

KEPLER'S THIRD LAW

(2) The derivation for elliptical orbits is not quite so easy.

Recall Kepler's 2nd law, which we studied in Chapter 3; "equal areas in equal time";

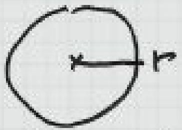
$$\frac{dA}{dt} = \frac{\ell}{2\mu}$$

- ANY CENTRAL FORCE
- CHAPTER 3
- CONSTANT

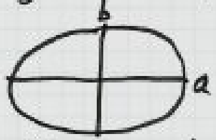
Thus $A = \dfrac{\ell\tau}{2\mu}$

WHAT IS THE AREA OF AN ELLIPSE?

What is the area of an ellipse?



$A = \pi r^2$



$A = \pi a b$

$\int dA = \int_{-a}^{a} 2y\,dx$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$= \int_{-a}^{a} 2b\sqrt{1 - x^2/a^2}\,dx \;= \pi a b$

So $\tau = \frac{2\mu}{\ell}\pi a b$

Recall: $a = \frac{\lambda}{1-\epsilon^2}$ and $b = \frac{\lambda}{\sqrt{1-\epsilon^2}}$

$$\tau = \frac{2\pi \mu a a \sqrt{1-\epsilon^2}}{\sqrt{\mu K a (1-\epsilon^2)}}$$

$$\tau = 2\pi\sqrt{\frac{a^3}{GM}}$$

$\tau^2 = \frac{4\pi^2 a^3}{(GM)^2}$ same as for a circular orbit! $(r \to a)$

• ***Section 8.6. Bounded Kepler Orbits***

We have been considering bounded Kepler orbits. These have energy $E < 0$.
The orbits are ellipses with eccentricity ε in the range $0 \leq \varepsilon < 1$.
(A circular orbit has $\varepsilon = 0$.)

Equations

$$r(\varphi) = \frac{\lambda}{1 + \varepsilon \cos \varphi}$$

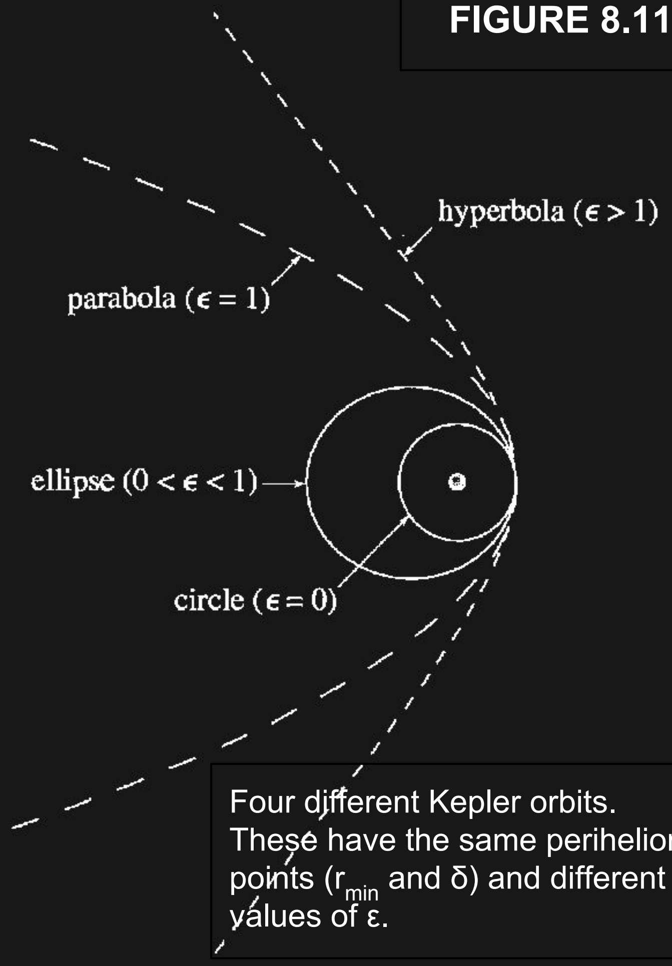
$$r_{min} = \lambda / (1+\varepsilon)$$

$$0 \leq \varepsilon < 1$$

$$\lambda = l^2 / (K\mu) \quad \text{and} \quad l = \mu r^2 \dot{\varphi}$$

$$E = \tfrac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} - \frac{K}{r}$$

FIGURE 8.11



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

Section 8.7. Unbounded Kepler Orbits

Now consider orbits with $E \geq 0$.

We can reuse some of the equations that we had before; they are valid for either
$E < 0$ or $E > 0$.

Equations

$$r(\varphi) = \frac{\lambda}{1 + \varepsilon \cos \varphi}$$

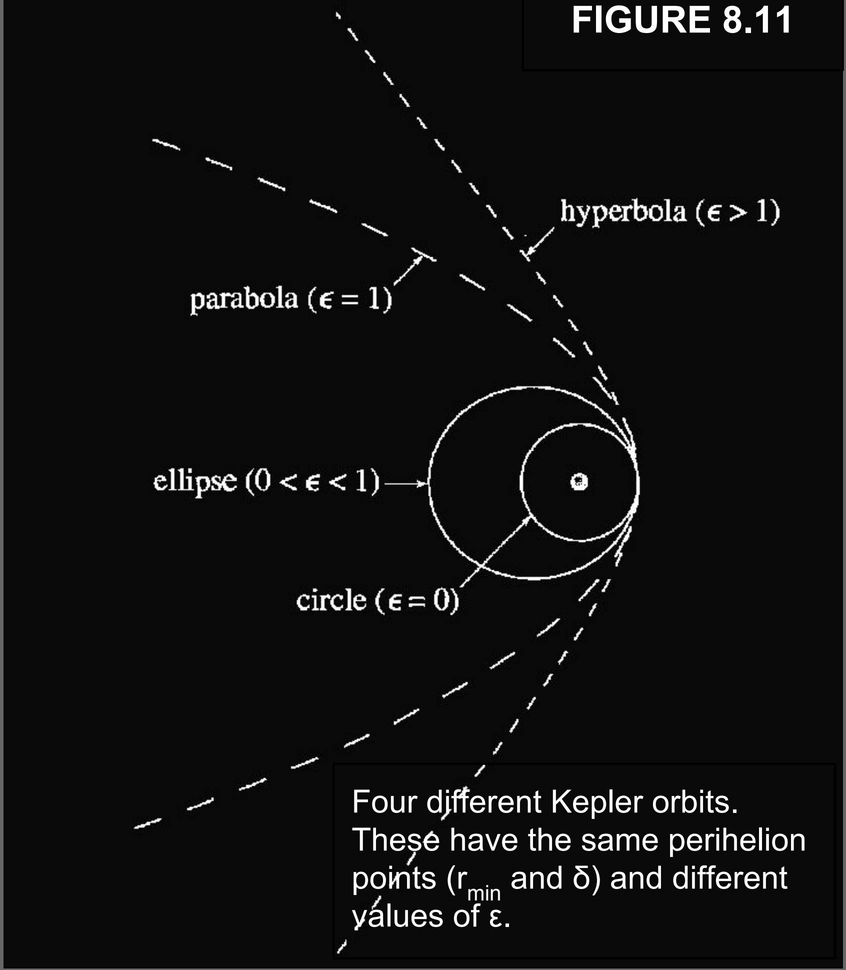
$$r_{min} = \lambda / (1+\varepsilon)$$

$\varepsilon \geq 1$

$$\lambda = l^2 / (K\mu) \quad \text{and} \quad l = \mu r^2 \dot{\varphi}$$

$$E = \tfrac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} - \frac{K}{r}$$

FIGURE 8.11



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

Parabolic orbits have $\varepsilon = 1$.

$$r(\varphi) = \frac{\lambda}{1 + \cos\varphi}$$

Why is this a parabola?

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

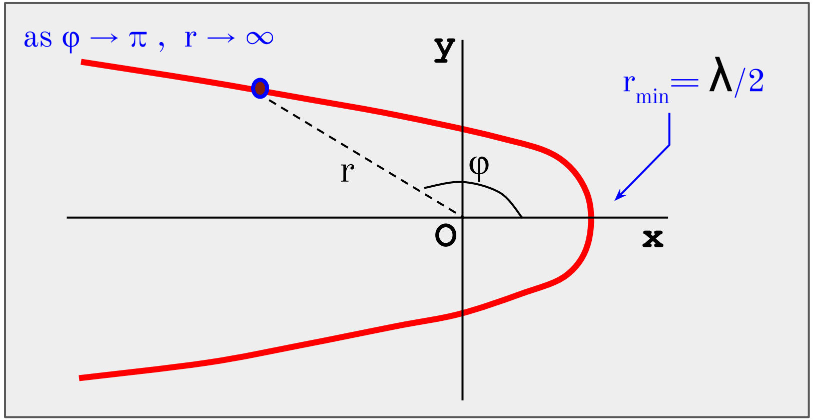
$$(1 + \cos\varphi)\, r = r + x = \lambda$$

$$r^2 = x^2 + y^2 = (\lambda - x)^2 = \lambda^2 - 2\lambda x + x^2$$

$$x = (\lambda^2 - y^2)/(2\lambda)$$

the eq. for a parabola

Or, $y = \pm\,\mathrm{sqrt}\,(\lambda^2 - 2\lambda x)$



The energy for the parabolic orbit

$$E = \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{K}{r}$$

$$\therefore E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{K}{r_{min}} \quad (\dot{r} = 0 \text{ at } r = r_{min})$$

Recall $\lambda = \dfrac{\ell^2}{K\mu}$ and $r_{min} = \dfrac{\lambda}{1+\epsilon} = \dfrac{\lambda}{2}$ PARABOLA

$$\therefore E = \frac{1}{r_{min}}\left(\frac{\ell^2}{2\mu r_{min}} - K\right) = \frac{1}{r_{min}}\left(\frac{\ell^2}{2\mu}\,\frac{2K\mu}{\ell^2} - K\right)$$
$$= 0$$

$E = 0$

Hyperbolic orbits have ε > 1

$$r(\varphi) = \frac{\lambda}{1 + \varepsilon \cos\varphi}$$

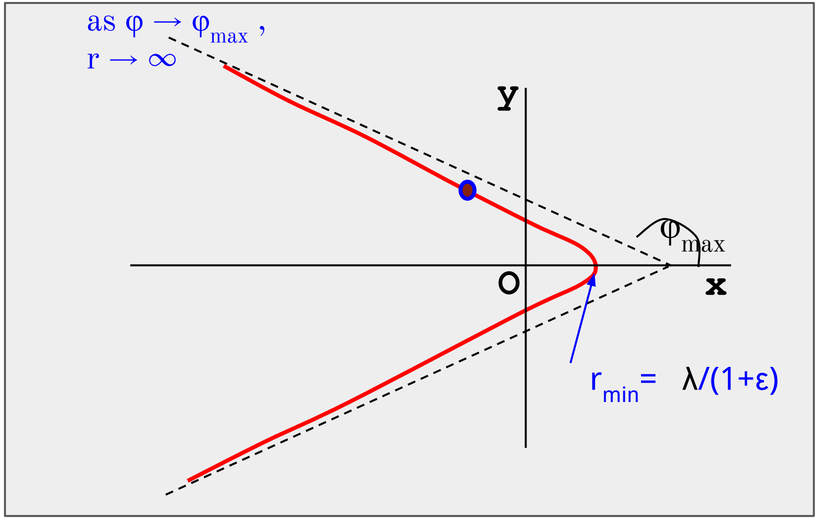
Why is this a hyperbola?

$$x = r \cos \varphi$$
$$y = r \sin \varphi$$

Note that $r \to \infty$ as $\varphi \to \varphi_{max}$

where $$\cos \varphi_{max} = -\frac{1}{\varepsilon}$$

which requires ε > 1



The energy for a hyperbolic orbit.

$$E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{K}{r_{min}} \quad (\dot r = 0)$$

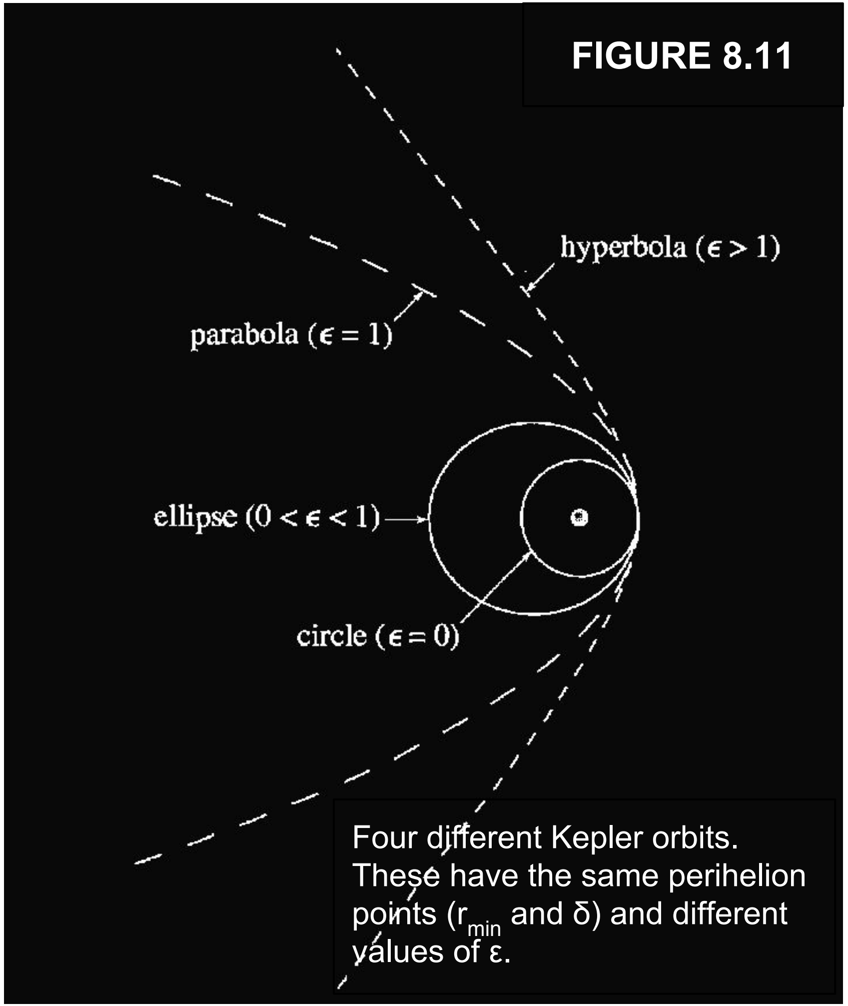
Recall $\lambda = \frac{\ell^2}{K\mu}$ and $r_{min} = \frac{\lambda}{1+\epsilon}$

$\Rightarrow r_{min} = \frac{\ell^2}{K\mu(1+\epsilon)}$

$$E = \frac{K^2\mu}{2\ell^2}(\epsilon^2 - 1)$$

So $E > 0 \quad (\epsilon > 1)$

E > 0



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

- Given the position and velocity vectors at one point on the orbit, the constants of motion ℓ and E are determined.
- The sign of E determines the curve:

$E < 0$	bounded	elliptical
$E = 0$	unbounded	parabolic
$E > 0$	unbounded	hyperbolic

- Given ℓ and E, the geometric parameters are determined; e.g., $\{r_{min}, r_{max}, \varepsilon\}$.
- These are the Kepler orbits in space; but what about the time dependence?

Exam 3 is Monday.
Homework assignment #14 is due next Friday.