EQUATION SHEETS FOR THE FINAL EXAM

Principal Definitions

We consider two-body central force problems.

Relative Coordinate and Reduced Mass

When written in terms of the ***relative coordinate***

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad (8.4)$$

and the CM coordinate **R**, the two-body problem is reduced to two parts:
(i) the problem of a free particle with mass $M = m_1 + m_2$ and position **R** ;
and (ii) the problem of a particle with mass μ = the ***reduced mass,***

$$\mu = m_1\, m_2 / (m_1 + m_2). \qquad (8.11)$$

The equivalent one-dimensional problem

The motion of the relative coordinate, with given angluar momentum l, is equivalent to the motion of a particle in one (radial) direction ($0 < r < \infty$) , with mass μ and ***effective potential energy***

$$U_{\text{eff}}(r) = U(r) + U_{\text{cf}}(r) = U(r) + l\hat{\ }2 \big/ (2\,\mu\, r^2)\ ; \qquad (8.30)$$

$U_{\text{cf}}(r)$ is called the ***centrifugal potential energy***.

The Transformed Radial Equation

With the change of variables from r to u = 1/r and elimination of t in favor of ϕ, the equation of the one-dimensional radial motion becomes

$$u''(\phi) = -\,u(\phi) - (\mu / l^2)\ F / u(\phi)^2\ . \qquad (8.41)$$

The Kepler Orbits

For a planet or comet, the force is $F = G\, m_1\, m_2 / r^2 = \gamma / r^2$;
and then the solution of (8.41) is

$r(\phi) =$ (8.49)

where $c = l^2 / (\gamma\mu)$ and ϵ is related to the energy by

$E =$ (8.58)

This ***Kepler orbit*** is an ellipse if ϵ is less than 1; ϵ is the eccentricity of the ellipse.

Other Relations for Kepler Orbits

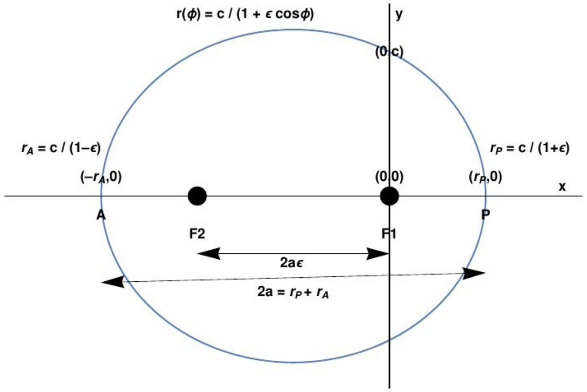
Let a = semimajor axios and ϵ = eccentricity.
Then $c = a\,(1 - \epsilon^2)$.
The angular momentum is l = SQRT[$c\,\gamma\,\mu$] = SQRT[$a\,\gamma\,\mu\,(1 - \epsilon^2)$]
The energy is $E = -\gamma / (2a)$.
The period of revolution is τ where $\tau^2 = (4\,\pi^2\, a^3) / (GM)$.

Ellipse Geometry

The figure below defines various geometrical factors for an ellipse.
a = semimajor axis; ϵ = eccentricity ;
c = semilatus rectum = $a\,(1 - \epsilon^2)$;
A = aphelion and r_A = aphelion distance ;
P = perihelion and r_P = perihelion distance.

In[458]:= fig

Out[458]=



Section 8.8.

Changes of Orbit

Read Section 8.8.

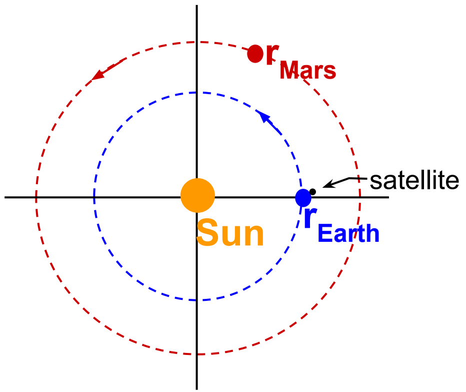
Here's a problem for NASA:
how to send a satellite from Earth to another planet, say Mars.

The satellite is too small to carry much fuel, so most of the trip must be done in "free fall", i.e., without rocket power.

I.e., the satellite moves along a Keplerian orbit, starting at Earth and ending at the other planet.

The initial conditions for the satellite are $r(0) = r_{Earth}$ and $v(0) = v_{Earth}$.

As a first approximation we can approximate the orbits of *Earth and Mars* as circular.



Now send the satellite to Mars, and ...

- hit the moving target
- use the minimum amount of rocket fuel

Hohmann transfer orbits

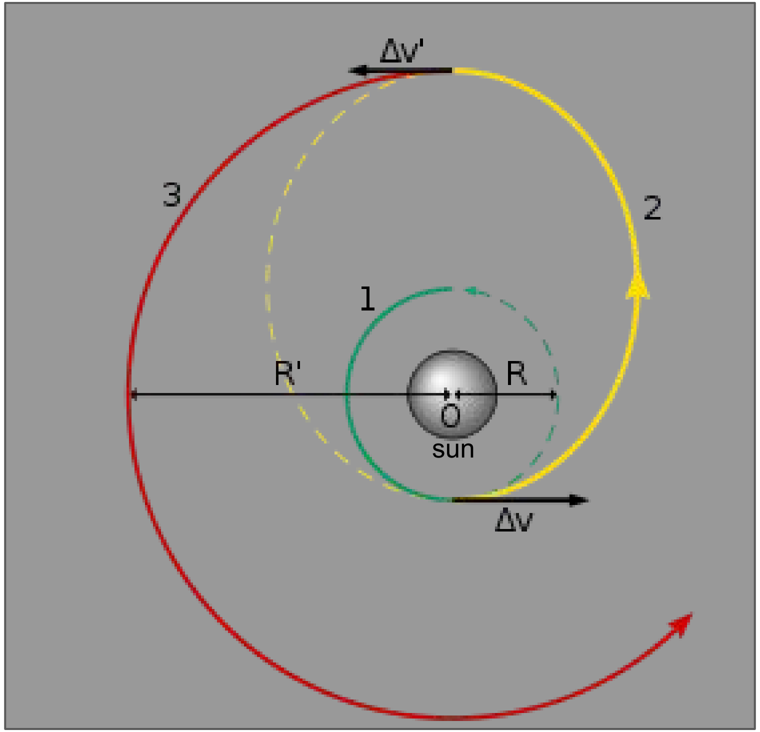
From Wikipedia, the free encyclopedia

In orbital mechanics, the **Hohmann transfer orbit** is an elliptical orbit used to transfer between two circular orbits of different radii in the same plane.

The orbital maneuver to perform the Hohmann transfer uses two engine impulses, one to move a spacecraft onto the transfer orbit and a second to move off it.

This maneuver was named after Walter Hohmann, the German scientist who published a description of it in his 1925 book *Die Erreichbarkeit der Himmelskörper* ("The Accessibility of Celestial Bodies")

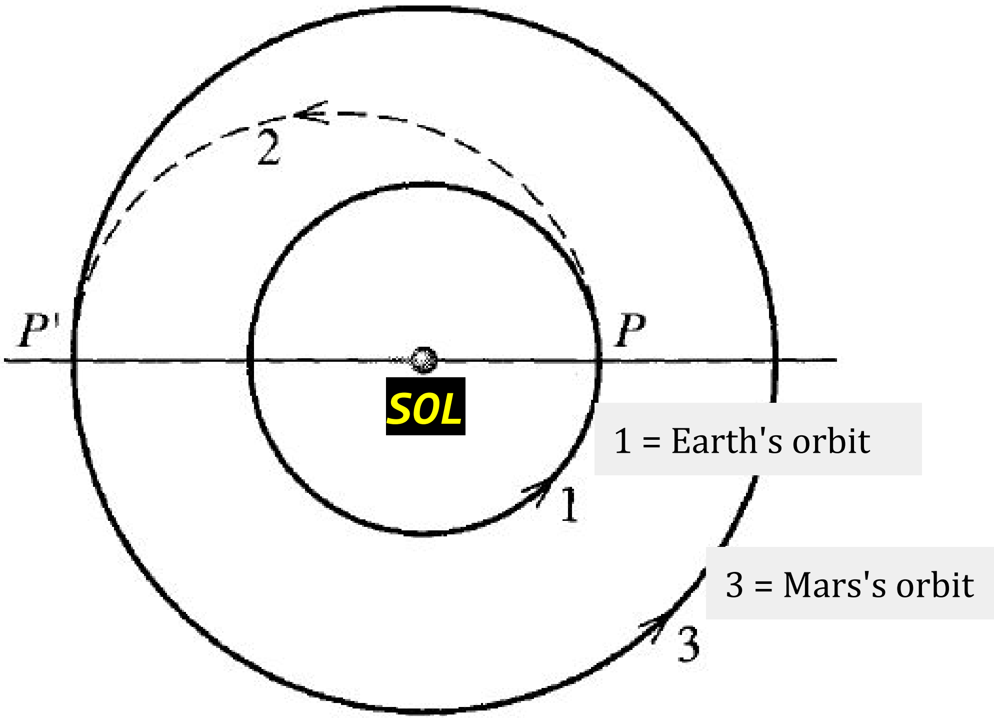
Hohmann was influenced in part by the German science fiction author Kurd Lasswitz and his 1897 book *Two Planets*.

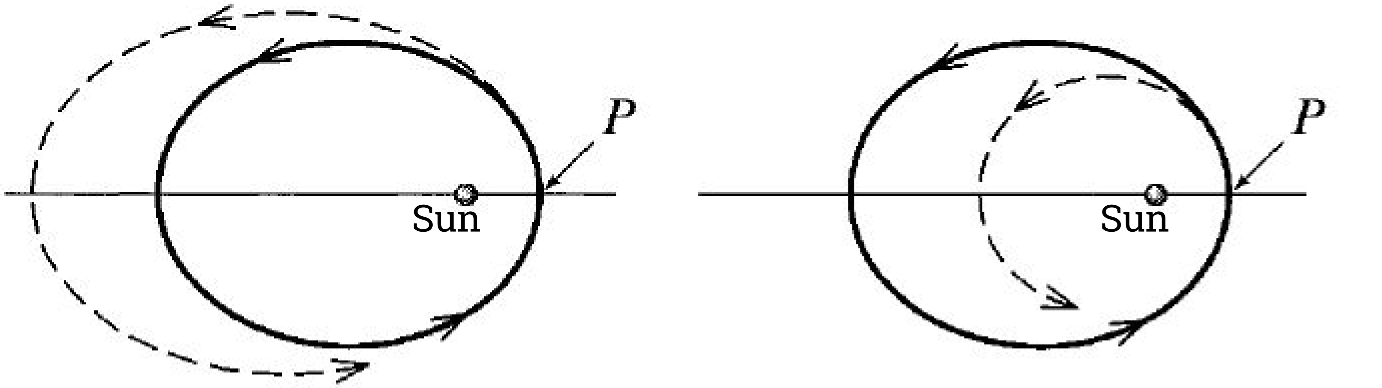


Hohmann transfer orbit, labelled 2, from a low orbit (1) to a higher orbit (3).

Sending a satellite to Mars :

- ***The transfer orbit is ½ of an ellipse; the dashed curve.***
- Perihelion (P) is at $r = r_1 = r_{Earth}$.
- Aphelion (P') is at $r = r_3 = r_{Mars}$.
- At P the speed of the satellite is the same (approximately) as the speed of the Earth, v_E.
- At P' the speed of the satellite must be raised to the speed of Mars, v_M.
- Forward thrust at P puts the satellite into a higher-energy orbit (2).
- Forward thrust at P' puts the satellite into an even higher-energy orbit (3).





(a) Forward thrust (b) Backward thrust

Forward thrust puts the satellite into a higher orbit, i.e., with larger orbital energy.

Backward thrust puts the satellite into a lower orbit, i.e., with smaller orbital energy.

☒ Traveling from Earth to Mars; ☒ approximate the planet orbits by circles; ☒ calculate r and v …

1. at P before firing the rocket

   {r [AU] , v [AU/y] , v [km/s] } =

2. at P just after firing the rocket

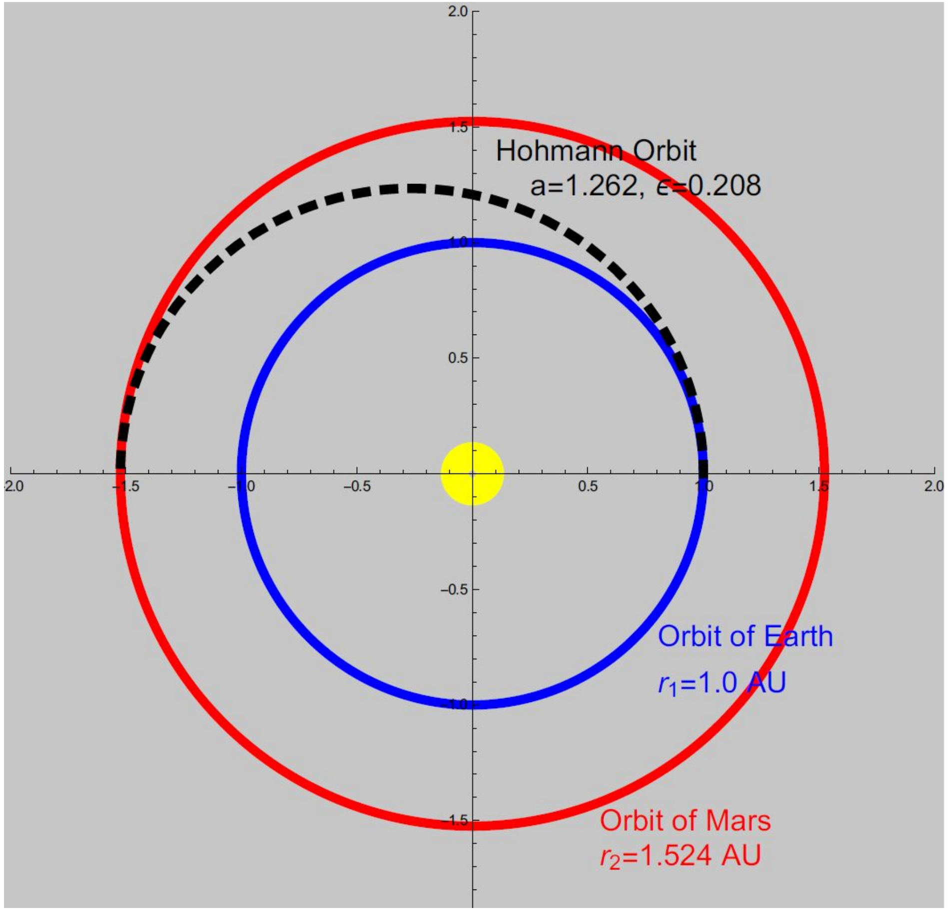
   {r [AU] , v [AU/y] , v [km/s] } =

3. at P' just before firing the rocket

   {r [AU] , v [AU/y] , v [km/s] } =

4. at P' after firing the rocket

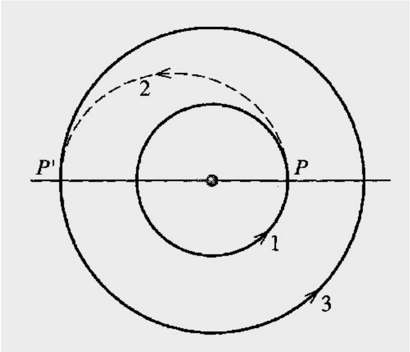
   {r [AU] , v [AU/y] , v [km/s] } =



Out[68]//TableForm=

1. Uau	$\frac{6.28319\ \text{Uau}}{\text{Uy}}$	$\frac{29.7857\ \text{Ukm}}{\text{Usec}}$
1. Uau	$\frac{6.90467\ \text{Uau}}{\text{Uy}}$	$\frac{32.7318\ \text{Ukm}}{\text{Usec}}$
1.524 Uau	$\frac{4.53062\ \text{Uau}}{\text{Uy}}$	$\frac{21.4776\ \text{Ukm}}{\text{Usec}}$
1.524 Uau	$\frac{5.08964\ \text{Uau}}{\text{Uy}}$	$\frac{24.1276\ \text{Ukm}}{\text{Usec}}$

Calculate the travel time to go from Earth to Mars on the Hohmann transfer orbit.



$r_1 = 149.5 \times 10^6$ km

$r_3 = 227.9 \times 10^6$ km

Recall Kepler's third law of planetary motion.

$$\tau^2 = \frac{4\pi^2 a^3}{GM} \quad ; \quad \text{or} \quad \tau^2 \propto a^3$$

For the transfer orbit (dashed curve 2)

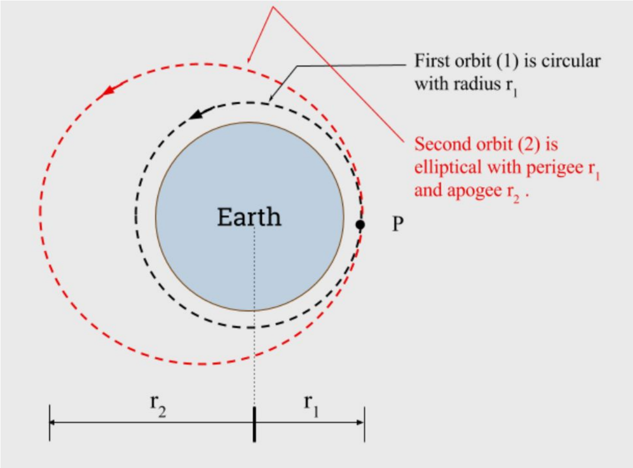
$$2a = r_1 + r_3 \quad \mapsto \quad a = 188.7 \times 10^6 \text{ km}$$

The calculation ...

The travel time is $\frac{1}{2}\,\tau$ = 259 days **= 8½ months .**

Another example of change of orbit (Taylor p 317)

NASA wants to put a satellite in an elliptical orbit (2) *around the Earth.* First they arrange the satellite in a circular orbit (1). Then they apply a short impulsive tangential thrust at the point P to put the satellite into the elliptical orbit (2). Given r_1 and r_2 calculate the *increase of the velocity* that must be supplied by the impulsive thrust.



Study Section 8.8 for the final exam.

https://en.wikipedia.org/wiki/Gravity_assist