Homework Problems due Friday Feb 24

total points = 18

Problem 25. Mandl and Shaw problem 4.1

Start with Eq. (4.53b), and derive the E.T.. aC. R.

Equation (4.53b) $\{\psi(x), \bar{\psi}(y)\} = i S(x-y)$,

where

$$S(x) = S^+(x) + S^-(x) = \left(i\not{\partial} + \frac{mc}{\hbar}\right)\Delta(x).$$

↑ Set $\hbar = 1$ and $c = 1$

Recall from Chapter 3,

$$\Delta(x) = \frac{-c}{(2\pi)^3}\int \frac{d^3k}{\omega_{\vec{k}}} \sin(k_\mu x^\mu).$$

So

$$S(x) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{\vec{k}^2+m^2}}\left\{\left(i\gamma^0\sqrt{\vec{k}^2+m^2} - i\vec{\gamma}\cdot\vec{k}\right)\cos(k_\mu x^\mu) + m\sin(k_\mu x^\mu)\right\}$$

Now set $x^0 = 0$.

$$S(0,\vec{x}) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{\vec{k}^2+m^2}}\left\{\left(i\gamma^0\underbrace{\sqrt{\vec{k}^2+m^2}}_{\text{even}} - \underbrace{i\vec{\gamma}\cdot\vec{k}}_{\text{odd}}\right)\frac{1}{2}\underbrace{\left(e^{i\vec{k}\cdot\vec{x}} + e^{-i\vec{k}\cdot\vec{x}}\right)}_{\text{even}} + \frac{m}{2i}\underbrace{\left(e^{i\vec{k}\cdot\vec{x}} - e^{-i\vec{k}\cdot\vec{x}}\right)}_{\text{odd}}\right\}$$

Where the integrand is an odd function of $\vec{k}$, the integral is 0. Thus

$$S(0,\vec{x}) = \frac{-1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{\vec{k}^2+m^2}}\, i\gamma^0\sqrt{\vec{k}^2+m^2}\, e^{i\vec{k}\cdot\vec{x}}$$

$$= -i\gamma^0\delta^3(\vec{x}).$$

2 points

$$\{\psi(x), \bar{\psi}(y)\}\Big|_{x^0=y^0} = i S(0, \vec{x}-\vec{y}) = \gamma^0\delta^3(\vec{x}-\vec{y}).$$

Problem 26. Mandl and Shaw problem 4.2.

- S(x) is a solution of the homogeneous Dirac equation.

$$S(x) = \frac{-\hbar}{(2\pi\hbar)^4}\int_{C^+} d^4p\, e^{-ip\cdot x/\hbar}\,\frac{\not{p}+mc}{p^2-m^2c^2} - \frac{\hbar}{(2\pi\hbar)^4}\int_{C^-} e^{-ip\cdot x/\hbar}\,\frac{\not{p}+m}{p^2-m^2}\,d^4p$$

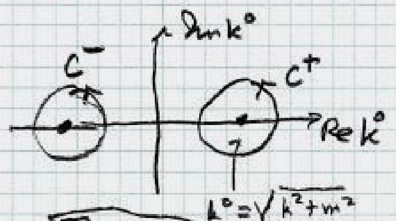

FIGURE 3.1

Set $\hbar = 1$ and $c = 1$.

(We can let the radii → 0.)

$$S(x) = \frac{-1}{(2\pi)^4}\int_{C^+} d^4p\, e^{-ip\cdot x}\,\frac{\not{p}+m}{p^2-m^2} - \frac{1}{(2\pi)^4}\int_{C^-} d^4p\, e^{-ip\cdot x}\,\frac{\not{p}+m}{p^2-m^2}$$

Now

$$(i\not{\partial}-m)\,S(x) = \frac{-1}{(2\pi)^4}\int_{C^+} d^4p\, e^{-ip\cdot x}\,\frac{(\not{p}-m)(\not{p}+m)}{p^2-m^2} + \text{same for } C^-$$

$$= \frac{-1}{(2\pi)^4}\int_{C^+} d^4k\, e^{-ip\cdot x} + \text{same for } C^-$$

$= 0$ because there is no pole in the k^0 plane $+ 0$

$= 0$ (homogeneous Dirac equation)

2 points

- $S_F(x)$ is a solution of an inhomogeneous Dirac equation.

$$S_F(x) = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x}\,\frac{\not{p}+m}{p^2-m^2+i\epsilon}$$

$$(i\not{\partial}-m)\,S_F = \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x}\,\frac{(\not{p}-m)(\not{p}+m)}{p^2-m^2+i\epsilon}$$

$$= \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip\cdot x} = \delta^4(x)$$ (inhomogeneous Dirac equation)

2 points

Problem 27. Mandl and Shaw problem 4.3.

Define the charge-current density operator,

$$s^\mu(x) = -\, e\, c\, \bar{\psi}(x)\, \gamma^\mu\, \psi(x)\ .$$

Show that microcausality is obeyed.

Set $\hbar=1$ and $c=1$.

Current density $s^\mu(x) = -e\,\bar\psi(x)\gamma^\mu\psi(x)$

Calculate

$[s^\mu(x), s^\nu(y)]$ for $(x-y)^2 = (x^0-y^0)^2-(\vec{x}-\vec{y})^2 < 0$.

Since x^μ and y^μ have spacelike separation, there exists a frame of reference in which $x^0 = y^0$. Also $|\vec{x}-\vec{y}|>0$ so that $(x-y)^2<0$.

By Lorentz covariance we can calculate the commutator in that frame of reference.

$$[s^\mu(x), s^\nu(y)] = e^2\left[\bar\psi(x)\gamma^\mu\psi(x),\ \bar\psi(y)\gamma^\nu\psi(y)\right]$$

$$= e^2\bar\psi(x)\gamma^\mu\left[\psi(x), \bar\psi(y)\gamma^\nu\psi(y)\right] + e^2\left[\bar\psi(x), \bar\psi(y)\gamma^\nu\psi(y)\right]\gamma^\mu\psi(x)$$

$$= e^2\bar\psi(x)\gamma^\mu\left(\{\psi(x),\bar\psi(y)\}\gamma^\nu\psi(y) - \bar\psi(y)\gamma^\nu\{\psi(x),\psi(y)\}\right)$$

$$+\, e^2\left(\{\bar\psi(x),\bar\psi(y)\}\gamma^\nu\psi(y) - \bar\psi(y)\gamma^\nu\{\bar\psi(x),\psi(y)\}\right)\gamma^\mu\psi(x)$$

Since $x^0 = y^0$ we can use the E.T.A.C.R.
(See Problem 4.1)

$$[s^\mu(x), s^\nu(y)] = e^2\bar\psi(x)\gamma^\mu\gamma^0\delta^3(\vec{x}-\vec{y})\gamma^\nu\psi(y)$$
$$-\, e^2\bar\psi(y)\gamma^\nu(\gamma^0)^T\delta^3(\vec{y}-\vec{x})\gamma^\mu\psi(x)$$

$= 0$ because $|\vec{x}-\vec{y}|>0$ so $\delta^3(x-y)=0$.

2 points

Problem 28. Mandl and Shaw problem 4.4.
Theorem. If we impose anticommutation relations for the Klein-Gordon field, then microcausality is violated.

Suppose $\phi(x) = \sum_{\vec{k}} \left(\frac{1}{2V\omega}\right)^{1/2} \left(a_{\vec{k}} e^{-ik\cdot x} + a_{k}^{+} e^{ik\cdot x}\right)$
with

$\{a_{\vec{k}}, a^{+}_{\vec{k}'}\} = \delta_{\vec{k}\vec{k}'}$ and $\{a_{\vec{k}}, a_{\vec{k}'}\} = 0 = \{a^{+}_{\vec{k}}, a^{+}_{\vec{k}'}\}$

Let x^μ and y^μ have spacelike separation;
i.e. $(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2 < 0$.
We can use a frame of reference in which
$x^0 - y^0 = 0$ and $|\vec{x}-\vec{y}| > 0$. W.L.O.G. let $x^0 = y^0 = 0$.

- $$\{\phi(x), \phi(y)\} = \sum_{\vec{k}} \sum_{\vec{k}'} \frac{1}{2V\sqrt{\omega\omega'}} \{ a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^{+}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}, a_{\vec{k}'} e^{i\vec{k}'\cdot\vec{y}} + a^{+}_{\vec{k}'} e^{-i\vec{k}'\cdot\vec{y}} \}$$

$$= \sum_{\vec{k},\vec{k}'} \frac{1}{2V\omega} \delta_{\vec{k},\vec{k}'} \left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} 2\cos[\vec{k}\cdot(\vec{x}-\vec{y})] \neq 0$$

2 points

- $$[\phi(x), \phi(y)] = \sum_{\vec{k}} \sum_{\vec{k}'} \frac{1}{2V\sqrt{\omega\omega'}} \left[a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^{+}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}, a_{\vec{k}'} e^{i\vec{k}'\cdot\vec{y}} + a^{+}_{\vec{k}'} e^{-i\vec{k}'\cdot y}\right]$$

$$= \sum_{\vec{k}} \sum_{\vec{k}'} \frac{1}{2V\sqrt{\omega\omega'}} [a_{\vec{k}} a_{\vec{k}'} - a_{\vec{k}'} a_{\vec{k}}]\, e^{i\vec{k}x} e^{i\vec{k}'y} + 3 \text{ other terms}$$

$$= \sum_{\vec{k}} \sum_{\vec{k}'} \frac{1}{2V\sqrt{\omega\omega'}} a_{\vec{k}} a_{\vec{k}'} \left[e^{i\vec{k}\cdot x} e^{i\vec{k}'\cdot\vec{y}} - e^{i\vec{k}'\cdot x} e^{i\vec{k}\cdot\vec{y}}\right] + 3\,OT$$

which is obviously not zero because this term (only) can annihilate 2 particles.

2 points

$$\langle 0 | [\phi(x), \phi(y)] | \vec{k}_1 \vec{k}_2\rangle = \frac{1}{2V\sqrt{\omega_1\omega_2}} \left[e^{i\vec{k}_1\cdot\vec{x}} e^{i\vec{k}_2\cdot\vec{y}} - e^{i\vec{k}_2\cdot\vec{x}} e^{i\vec{k}_1\cdot\vec{y}} - e^{i\vec{k}_2\cdot\vec{x}} e^{i\vec{k}_1\cdot\vec{x}} + e^{i\vec{k}_1\cdot\vec{x}} e^{i\vec{k}_2\cdot\vec{y}}\right]$$

$\neq 0$

Problem 29. Mandl and Shaw problem 4.5.
Chiral phase transformations for the Dirac field.

The chiral transformation is

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x)$$

$$\psi^\dagger(x) \to \psi'^\dagger(x) = \psi^\dagger(x)\, e^{-i\alpha\gamma_5}$$

where α = real and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Properties of γ_5 : $\{\gamma^\mu, \gamma_5\} = 0$; $\gamma_5^2 = 1$; $\gamma_5^\dagger = \gamma_5$.

Transformation of the Lagrangian density

$$\mathcal{L} = \bar\psi\,(i\gamma^\mu\partial_\mu - m)\,\psi$$

$$\mathcal{L}' = \bar\psi'\,(i\gamma^\mu\partial_\mu - m)\,\psi' = \psi'^\dagger\gamma^0\,(i\gamma^\mu\partial_\mu - m)\,\psi'$$

$$= \psi^\dagger\, e^{-i\alpha\gamma_5}\gamma^0\,(i\gamma^\mu\partial_\mu - m)\, e^{i\alpha\gamma_5}\psi$$

Note that $e^{-i\alpha\gamma_5}\gamma^0 = \gamma^0 e^{+i\alpha\gamma_5}$ b/c $\{\gamma_5, \gamma^0\} = 0$

Also $e^{i\alpha\gamma_5}\gamma^\mu = \gamma^\mu e^{-i\alpha\gamma_5}$ b/c $\{\gamma_5, \gamma^\mu\} = 0$

$$\mathcal{L}' = \psi^\dagger\gamma^0\left(i\gamma^\mu e^{-i\alpha\gamma_5}\partial_\mu - m e^{i\alpha\gamma_5}\right) e^{i\alpha\gamma_5}\psi$$

$$= \bar\psi\left(i\gamma^\mu\partial_\mu - m\, e^{2i\alpha\gamma_5}\right)\psi$$

$\mathcal{L}' = \mathcal{L}$ if $m = 0$.

2 points

Problem 29. Mandl and Shaw problem 4.5.
Chiral phase transformations for the Dirac field.

Consider the axial current $J_A^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

Calculate

$$\partial_\mu J_A^\mu = (\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma^\mu\gamma_5(\partial_\mu\psi)$$

$$= (\partial_\mu\bar{\psi})\gamma^\mu\gamma_5\psi \quad - \bar{\psi}\gamma_5\gamma^\mu(\partial_\mu\psi)$$

The Dirac equation $(i\gamma\cdot\partial - m)\psi = 0 \Rightarrow \gamma^\mu\partial_\mu\psi = -im\psi$

Adjoint : $(\partial_\mu\psi^\dagger)(\gamma^\mu)^\dagger = im\psi^\dagger$

$$(\partial_\mu\psi^\dagger)\gamma^0\gamma^\mu\gamma^0 = im\,\psi^\dagger\gamma^0\gamma^0$$

cancel

$$(\partial_\mu\bar{\psi})\gamma^\mu = im\bar{\psi}$$

Thus

$$\partial_\mu J_A^\mu = im\bar{\psi}\gamma_5\psi - \bar{\psi}\gamma_5(-im\psi) = 2im\bar{\psi}\gamma_5\psi$$

$\therefore$ If $m=0$ then J_A^μ is conserved.

2 points

Define $\psi_L = \frac{1}{2}(1-\gamma_5)\psi$ and $\psi_R = \frac{1}{2}(1+\gamma_5)\psi$

Field equations —

$$i\gamma^\mu\partial_\mu\psi_L = i\gamma^\mu\frac{1}{2}(1-\gamma_5)\partial_\mu\psi = i\frac{1}{2}(1+\gamma_5)\gamma^\mu\partial_\mu\psi$$

$$= i\frac{1}{2}(1+\gamma_5)(-im)\psi = m\psi_R$$

and similarly

$$i\gamma^\mu\partial_\mu\psi_R = m\psi_R$$

If $m=0$ then ψ_L and ψ_R decouple.

2 points