Mandl and Shaw reading assignments

Chapter 2 Lagrangian Field Theory

2.1 Relativistic notation

2.2 Classical Lagrangian field theory

2.3 Quantized Lagrangian field theory

2.4 Symmetries and conservation laws

Problems; 2.1 2.2 2.3 2.4 2.5

Chapter 3 The Klein-Gordon Field

3.1 The real Klein-Gordon field

3.2 The complex Klein-Gordon field

3.3 Covariant commutation relations

3.4 The meson propagator

Problems; 3.1 3.2 3.3 3.4 3.5

Forget about particles.

<u>*What are the fields?*</u>

What equations govern the fields?

- ★ We always start with a classical field theory.
- ★ The field equations come from Lagrangian dynamics.

Mandl and Shaw, Chapter 2

Review of Lagrangian dynamics

For a single coordinate q(t) :

Lagrangian $L = L(q, dq/dt)$;

and Action $A = \int_{t1}^{t2} L(q, dq/dt)\, dt$.

The equation of motion for q(t) comes from the requirement that $\delta A = 0$ (with endpoints fixed); i.e., the action connecting initial and final points is an extremum. Now consider a variation $\delta q(t)$

$$\delta A = 0 = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}(\delta q) \right] dt$$

integrate by parts ; $\delta q = 0$ at t_1 and t_2

$$= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q \, dt$$

must be 0 for any variation $\delta q(t)$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$$

(Lagrange's equation)

Canonical momentum and the Hamiltonian

$$p = \frac{\partial L}{\partial \dot{q}}$$

$$H = p\dot{q} - L$$

which ~~must~~ be rewritten in terms of p and q

Example. A particle in a potential…

$$q(t) = \vec{x}(t)$$

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - V(\vec{x})$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = \frac{d}{dt}(m\dot{\vec{x}}) + \frac{\partial V}{\partial \vec{x}}$$

$$= 0$$

$$\ddot{\vec{x}} = \vec{F}/m \quad \text{where} \quad \vec{F} = -\partial V/\partial \vec{x}$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}}$$

$$H = p\dot{x} - L = \frac{p^2}{m} - \frac{p^2}{2m} + V(\vec{x})$$

$$H = \frac{p^2}{2m} + V(\vec{x})$$

… the same as classical particle dynamics.

Canonical Quantization (Dirac)

Rules to convert classical dynamics to a quantum theory:

- q and p become *operators;* they operate on the Hilbert space of physical states.
- $[\, q\, ,\, p\,] = i\,\hbar$
- H is the generator of translation in time.

Theorem.
H is the generator of translation in time for the quantum theory; *prove it for the Heisenberg picture.*
Suppose L = ½ M $(dq/dt)^2 - V(q)$.

$$H = \frac{p^2}{2m} + V(q)$$

- $\frac{i}{\hbar}[H,q] = \frac{i}{\hbar}\frac{1}{2m}[p^2, q]$
  $= \frac{i}{2m\hbar}\{p[p,q] + [p,q]p\}$
  $= \frac{i}{2m\hbar}(-i\hbar p\, 2) = \frac{p}{m} = \frac{dq}{dt}$ ✓
- $\frac{i}{\hbar}[H,p] = \frac{i}{\hbar}[V(q), p]$
  $= \frac{i}{\hbar}\, i\hbar \frac{\partial V}{\partial q} = -\frac{\partial V}{\partial q} = \frac{dp}{dt}$ ✓

Q.E.D.

So far, we have considered only one degree of freedom. Now consider a system with many degrees of freedom; $\{ q_i : i = 1\ 2\ 3 \ldots N \}$

For many degrees of freedom…

$$q(t) \longrightarrow Q(t) \equiv \{ q_i(t) ; i = 1\ 2\ 3 \ldots i \ldots N \}$$

- L = L(Q, dQ/dt) ⇒ N Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad \text{for } i = 1\ 2\ 3 \ldots N$$

- Canonical momenta … $p_i = \frac{\partial L}{\partial \dot{q}_i}$
- and Hamiltonian…

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - L$$

(which *must be* re-expressed in terms of $p_1 \ldots p_N$ and $q_1 \ldots q_N$.)

Classical field theory (suppress spin for now)

We replaced

$q(t) \longrightarrow \{ q_i(t) \; ; \; i \in Z \}$; *discrete*

Now replace

$q(t) \longrightarrow \{ \psi(\mathbf{x},t) \; ; \; \mathbf{x} \in R^3 \}$; *continuum*

♦ ♦ ♦

Might try $L = L(\psi(\mathbf{x},t), \; \partial\psi(\mathbf{x},t)/\partial t)$; *but no.*

$L = \int \mathcal{L}(\psi(\mathbf{x},t), \; \nabla\psi(\mathbf{x},t), \; \partial\psi(\mathbf{x},t)/\partial t) \, d^3x$

∴ Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\psi}(x)}\right) - \frac{\delta L}{\delta \psi(x)} = 0$$

(functional derivatives)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)}\right) - \frac{\partial \mathcal{L}}{\partial \psi(x)} + \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla\psi)} = 0$$

i.e. vary ψ but keep $\nabla\psi$ fixed.

(partial derivatives)

this is the *"classical field theory."*

- - - an example of continuum dynamics.

EXAMPLE: THE LAGRANGIAN FOR SCHROEDINGER WAVE MECHANICS

$$A = \int_{t_1}^{t_2} dt \int d^3x \left\{ -\frac{i\hbar}{2}\left(\frac{\partial\psi^*}{\partial t}\psi - \psi^*\frac{\partial\psi}{\partial t}\right) - \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi - V\psi^*\psi \right\}$$

Note: A is real, but $\psi(\mathbf{x})$ is complex.

We could write $\psi = R + i\,I$ and treat R and I as independent "generalized coordinates". Then require $\delta A = 0$ for any $\delta R(x)$ and $\delta I(x)$.

Easier, and equivalent, is to treat ψ and ψ^* as independent generalized coordinates, and require $\delta A = 0$ for independent variations of $\delta\psi$ and $\delta\psi^*$.

Lagrange's Equations

Vary A w.r.t. ψ^* keeping ψ fixed $\Rightarrow$

$$\delta A = \int_{t_1}^{t_2} \int d^3x \left\{ \frac{-i\hbar}{2} \left[\frac{\partial}{\partial t}(\delta\psi^*)\,\psi - \delta\psi^* \frac{\partial\psi}{\partial t} \right] - \frac{\hbar^2}{2m} \nabla\delta\psi^* \cdot \nabla\psi - V\,\delta\psi^*\,\psi \right\}$$

Integrate by parts; the surface terms are 0. (b/c $\delta\psi^* = 0$ at t_1 and t_2; and $\psi^* = 0$ at ∞)

$$\delta A = \int_{t_1}^{t_2} dt \int d^3x\, (\delta\psi^*) \left\{ \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} + \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - V\psi \right\}$$

$= 0$ for all variations $\delta\psi^*$.

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

Thus, the classical field equation is the Schroedinger equation.

The other Lagrange equation, from $\delta\psi(x)$ keeping ψ^* fixed, is

$$-i\hbar \frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*$$

so ψ^* = the complex conjugate of ψ.

Canonical momenta

$$\pi(\vec{x}) = \frac{\delta L}{\delta\dot{\psi}(\vec{x})} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = \frac{i\hbar}{2}\psi^*(x)$$

$$\pi^*(\vec{x}) = \frac{\delta L}{\delta\dot{\psi}^*(x)} = -\frac{i\hbar}{2}\psi(x)$$

The Hamiltonian

The Hamiltonian

$$H = \int d^3x\, (\pi\dot{\psi} + \pi^*\dot{\psi}^*) - L$$

$$= \int d^3x \left\{ \frac{i\hbar}{2}\psi^*\dot{\psi} - \frac{i\hbar}{2}\psi\dot{\psi}^* + \frac{i\hbar}{2}\dot{\psi}^*\psi - \frac{i\hbar}{2}\psi^*\dot{\psi} + \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi \right\}$$

$$= \int d^3x \left\{ \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi \right\}$$

just the same as in second quantization! (but this is still the classical field theory)

Quantization of the field

So far, this is the classical field theory. Now…

Dirac's canonical commutation relation $[\, q\, ,\, p\,] = i\hbar$ is valid for Hermitian operators q and p. We need to modify that (*because ψ is complex)* to

$$[\psi(x), \pi(x')] = \frac{i\hbar}{2}\,\delta^3(x-x') \quad \text{where } \pi = \frac{i\hbar}{2}\psi^\dagger$$

$$[\psi^\dagger(x), \pi^\dagger(x')] = \frac{i\hbar}{2}\,\delta^3(x-x') \quad \text{where } \pi^\dagger = -\frac{i\hbar}{2}\psi$$

Therefore

$$[\psi(x), \psi^\dagger(x')] = \delta^3(x-x')$$

$$[\psi(x), \psi(x')] = 0$$

which is just the same as in second quantization!

Replace $[\,,\,]$ by $\{\,,\,\}$ (anticommutators) for fermions.

Summary

$$[\, \psi(\mathbf{x})\, ,\, \psi(\mathbf{x}')\,] = 0$$

$$[\, \psi(\mathbf{x})\, ,\, \psi^\dagger(\mathbf{x}')\,] = \delta^3(\mathbf{x} - \mathbf{x}') \quad ;$$

or, use *anticommutators* for fermions;

$$H = \int d^3x \left\{ \frac{\hbar^2}{2m} \nabla\psi^\dagger \cdot \nabla\psi + V\psi^\dagger\psi \right\}$$

This is precisely the NRQFT that we considered last week, but with a 1-body potential V(**x**) and without a 2-body potential V_2(**x**,**y**).

Exercise: Figure out the Lagrangian that would include a 2-body potential. Hint: The Lagrangian must include a term quartic in the field.

Exercise: Verify that H is the generator of translation in time, in the quantum theory.

Homework Problems due Fri Feb 10

Problem 16.
Equal time commutation relations.
We have, in the Schroedinger picture,

$$[\, \psi(\mathbf{x}) \,,\, \psi^\dagger(\mathbf{x}') \,] = \delta^3(\mathbf{x} - \mathbf{x}') \quad ,$$

$$[\, \psi(\mathbf{x}) \,,\, \psi(\mathbf{x}') \,] = 0 \text{ , etc.}$$

(a) Show that in the Heisenberg picture, this commutation relation holds *at all equal times*.
(b) What is the commutation relation for *different* times?

Problem 17.
(a) Do problem 2.1 in Mandl and Shaw.
(b) Do problem 2.2 in Mandl and Shaw.
(c) Do problem 2.3 in Mandl and Shaw.