QED; and the Standard Model

We have calculated cross sections in lowest order perturbation theory.

Terminology: *Born approximation; tree diagrams.*

At this order of approximation QED (and the standard model) seem pretty good, compared to the data.

But now we want to compare theory and data to higher precision. So we need higher order calculations.

Higher order corrections should be small, because $e^2 = 4\pi/137$ is small.

When we calculate higher order contributions, we encounter *divergent integrals.*

There are two kinds of divergence – infrared (IR) and ultraviolet (UV) – which have different origins.

The IR divergences occur because of massless fields: the photon field in QED and the gluon fields in QCD.

The UV divergences occur because of point like interactions. In QED,

$$\pounds_{\text{int}}(x) = e\,\overline{\psi}(x)\,\gamma^{\mu}\,\psi(x)\,A_{\mu}(x)$$

which implies interactions with arbitrarily large momentum.

After careful considerations, *divergent quantities cancel …*

These topics are related to each other:

- ➔ Radiative corrections
- ➔ Divergences
- ➔ Renormalizations

⬇

Chapters 9 and 10

⬆

Today: *Bremsstrahlung* (Sec 8.8) and the *Infrared Divergence* (Sec 8.9).

Bremsstrahlung

(‘brems’=braking, ‘strahlung’=radiation)

In classical electrodynamics, when a charge undergoes acceleration, it radiates electromagnetic waves.

Now consider an electron scattering from an “infinitely massive charge”, Ze.

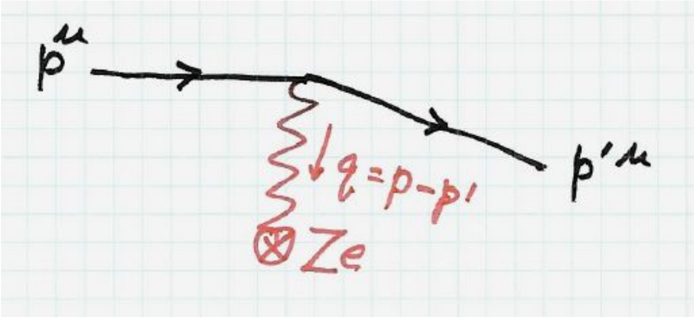
What is the radiation?

- *Mott scattering*

Suppose an electron scatters from a "heavy charge", Ze.

E.g., a U nucleus has $M \approx 238$ GeV >>> m_e .

We can treat the heavy charge in the limit $M \to \infty$. In its rest frame it can absorb any amount of momentum without recoiling; there is no energy transfer (*elastic scattering*).



$$\mathcal{M} = \bar{u}(p')\, ie\gamma^\mu\, u(p)\, \hat{A}_\mu(q)$$

$$\hat{A}_\mu(q) = \delta_{\mu 0} \frac{Ze}{|\vec{q}|^2}$$

$\Rightarrow$ the Mott cross section in the lab frame,

$$\frac{d\sigma}{d\Omega'} = \frac{2\alpha^2 Z^2}{|\vec{q}|^4}\left(E^2 + \vec{p}\cdot\vec{p}\,' + m^2\right)$$

Here $\vec{q} = \vec{p} - \vec{p}\,'$ and $|\vec{p}\,'| = |\vec{p}|$.

Or,

$$\frac{d\sigma}{d\Omega'} = \frac{\alpha^2 Z^2}{4E^2 v^4 \sin^4\frac{\theta}{2}}\left(1 - v^2 \sin^2\frac{\theta}{2}\right)$$

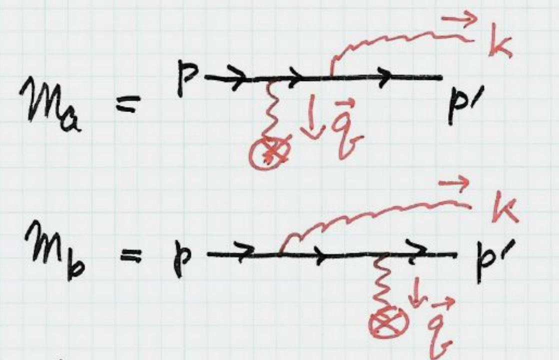
Note the resemblance to the Rutherford cross section

- **Bremsstrahlung** (Sections 8.8 & 8.9)

Now consider, in Q.E.D.,

$$\begin{array}{ccccccccc} e & + & Z & \rightarrow & e' & + & Z & + & \gamma \\ p^{\mu} & & & & p'^{\mu} & & & & k^{\mu} \end{array}$$

There are two Feynman diagrams



$$\vec{q} = \vec{p} - \vec{p}' - \vec{k}$$

We won't calculate the full cross section, but we need to identify some parts of it.

$\mathfrak{M}_a$

$$\mathcal{M}_a \propto S_F(p' + k) \propto \frac{1}{(p'+k)^2 - m^2}$$

$$\begin{aligned} D_a &= (p'+k)^2 - m^2 = 2p' \cdot k \\ &= 2(E'\omega - p'k\cos\theta_a) \\ &= 2\omega(E' - p'\cos\theta_a) \end{aligned}$$

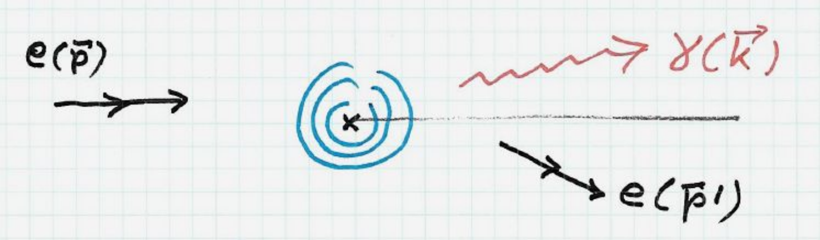
$\mathfrak{M}_b$

$$\mathcal{M}_b \propto S_F(p - k) \propto \frac{1}{(p-k)^2 - m^2}$$

$$\begin{aligned} D_b &= (p-k)^2 - m^2 = -2p \cdot k \\ &= -2[E\omega - pk\cos\theta_b] \\ &= -2\omega(E - p\cos\theta_b) \end{aligned}$$

- *The Bethe-Heitler cross section (1935)*

which they calculated from the Dirac equation and perturbation theory



Lorentz invariant phase space,

$$\frac{d^3k}{2\omega} = \frac{k^2 dk\, d\Omega_k}{2\omega} = \frac{1}{2}\, k\, dk\, d\Omega_k$$

$$\frac{d^3\sigma}{dk\, d\Omega_k\, d\Omega_{p'}} = \frac{\alpha Z^2 r_0^2}{\pi^2 q^4}\, \frac{p'}{p}$$

$$r_0 = \frac{e^2}{mc^2}$$

$$k \left\{ \frac{4E_2^2 - q^2}{D_b^2} (\vec{p} \times \hat{n})^2 + \frac{4E_1^2 - q^2}{D_a^2} (\vec{p}\,' \times \hat{n})^2 + \frac{2(4E_1E_2 - q^2)}{D_a D_b} (\vec{p} \times \hat{n}) \cdot (\vec{p}\,' \times \hat{n}) - \frac{2k^2 (\vec{q} \times \hat{n})^2}{D_a D_b} \right\}$$

where $\vec{k} = k\hat{n}$

where **k** = k **n** .

$\Rightarrow$ Do you see the Mott cross section?

$\Rightarrow$ $D_b = (p - k)^2 - m^2 = -2\,E\,k + 2\,\mathbf{p}\,.\,\mathbf{k}$

$\Rightarrow$ $D_a = (p' + k)^2 - m^2 = 2\,E'\,k - 2\,\mathbf{p}'.\,\mathbf{k}$

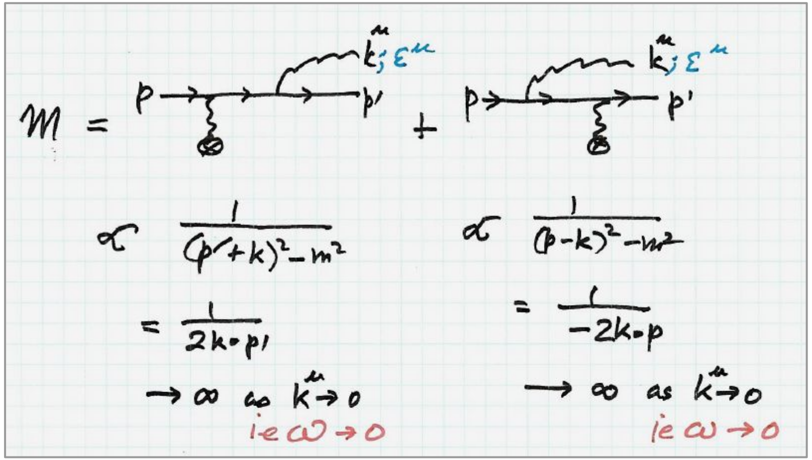
$\Rightarrow$ The cross section diverges as $k \to 0$.

(Interpretation?
Soft photons are infinitely likely?)

- *The origin of the "Infrared Catastrophe"*

There are IR divergences because the photon is massless.

The first sign of the IR divergences is the fact that $d\sigma_{Brems} \to \infty$ as $\omega \to 0$.



- *The discussion in Section 8.8.*

$$\mathcal{M} = -e^2\, \bar{u}(p')\, \gamma\cdot\hat{A}^\mu(q)\, u(p) \left[\frac{p'\cdot\epsilon}{p'\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right]$$

$$\mathcal{M} \underset{\omega\to 0}{\sim} -e\, \mathcal{M}_{Mott} \left(\frac{p'\cdot\epsilon}{p'\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right)$$

$$\frac{d\sigma}{d\Omega'} \underset{\omega\to 0}{\sim} \left(\frac{d\sigma}{d\Omega'}\right)_{Mott} \underbrace{\frac{\alpha}{2\pi}\left(\frac{p'\cdot\epsilon}{p'\cdot k} - \frac{p\cdot\epsilon}{p\cdot k}\right)^2}_{\text{"soft photon emission factor"}} \frac{d^3k}{\omega}$$

$$= \frac{\alpha}{2\pi}\, \frac{k^2\, dk\, d\Omega_k}{\omega} \left[\frac{p'\cdot\epsilon}{D_a/2} + \frac{p\cdot\epsilon}{D_b/2}\right]$$

$$\sim \frac{1}{\omega} \text{ as } \omega \to 0\,;$$

e.g. $p'\cdot k = E'\omega - p'k\cos\theta_a$
$= \omega\left[E' - p'\cos\theta_a\right]$

Why do $\mathfrak{M}_{Brems}$ and $d\sigma_{Brems}$ diverge as $\omega \to 0$? I.e., what is the *mathematical reason*? As $k^\mu \to 0$, the denominator of the electron propagator $\to 0$.

Comment.

The infrared problem is not a surprise, because *classical* electrodynamics has the same problem.

See J. D. Jackson, *Classical Electrodynamics*.

When a charged particle accelerates, it ***must*** radiate electromagnetic waves.

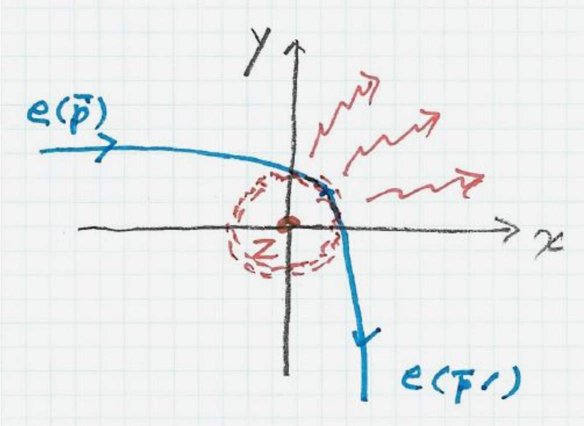


Figure 15.3 in Jackson…

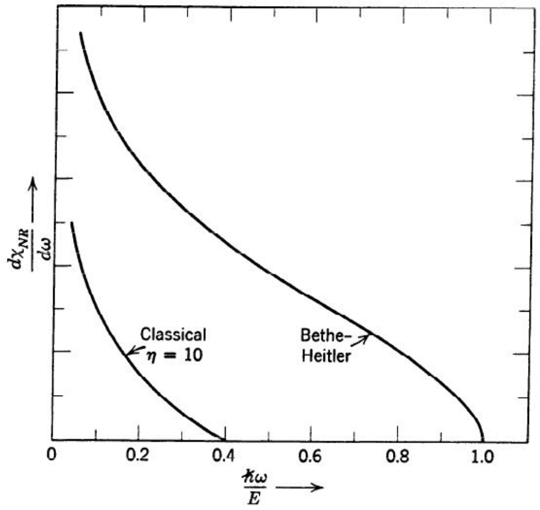


Figure 15.3 Radiation cross section (energy × area/unit frequency) for nonrelativistic Coulomb collisions as a function of frequency in units of the maximum frequency ($E/\hbar$). The classical spectrum is confined to very low frequencies. The curve marked "Bethe–Heitler" is the quantum-mechanical Born approximation result, i.e., (15.29) with $\lambda' = 1$.

In classical electrodynamics, the e.m. wave can be subdivided into smaller and smaller amplitudes; energy is not quantized. But in the quantum theory, there cannot be a fraction of a photon.

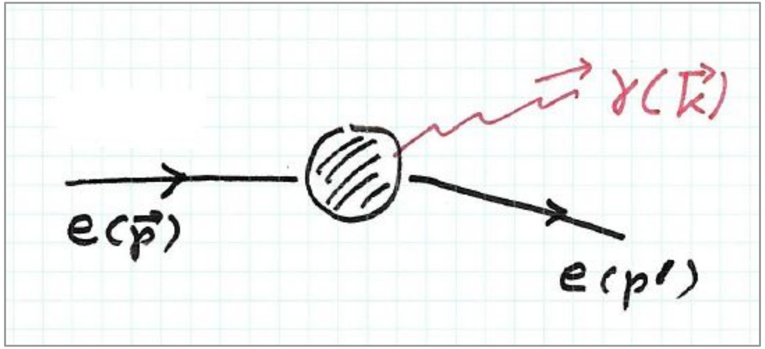
- *The solution of the infrared problem*

Bloch and Nordsieck (1937)

Think about a *real* experiment.
What can actually be measured?

A real detector will have a *photon energy resolution* $\equiv \Delta E$.

If $E_\gamma < \Delta E$ then the photon is unobservable.



We would observe two processes in the experiment:

(a) "elastic scattering"

$e + Z \rightarrow e' + Z$ + photons with $\omega < \Delta E$

(below the resolution)

This is not the *ideal* elastic cross section; it's the *actual* measurement.

(b) "Bremsstrahlung"

$e + Z \rightarrow e' + Z$ + photons with $\omega > \Delta E$

(above the resolution)

This is not the *ideal* radiation rate; it's the *actual* measurement.

- *(b) the actual measurement of Bremsstrahlung;*

i.e., with $E_\gamma > \Delta E$; only counting observable photons

$$\left(\frac{d\sigma}{d\Omega'_e}\right)_{(b)} = \int_{\Delta E}^{E_{max}} \frac{d^2\sigma}{dk\, d\Omega'_e}\, dk$$

$$\sim \left(\frac{d\sigma}{d\Omega'_e}\right)_{Mott} \alpha \int_{\Delta E}^{E_{max}} \frac{dk}{k}$$

assuming E_{max} is small enough to use the factorization

$$= \left(\frac{d\sigma}{d\Omega'_e}\right)_{Mott} \alpha \ln \frac{E_{max}}{\Delta E}$$

The result is divergent as $\Delta E \to 0$, but ONLY LOGARITHMICALLY.

E.g., Suppose $E_{max} = 1$ GeV and $\Delta E = 1$ eV.

Then $\left(\frac{d\sigma}{d\Omega'}\right)_{(b)} \sim \left(\frac{d\sigma}{d\Omega'}\right)_{Mott} \frac{1}{137} \ln 10^9 = \left(\frac{d\sigma}{d\Omega'}\right)_{Mott} \times 0.15$

i.e., the observed radiation rate is only a small correction to Mott scattering.

So there is no divergence in the prediction for the *observable* Bremsstrahlung cross section.

- *(a) Fine, but what about the actual measurement of "elastic scattering"?*

i.e., the process (a), which must include photon emission with $E_\gamma < \Delta E$ (unobservable soft photons)

That's where the IR divergence occurs now. Isn't that a real problem?

Yes, that is a real problem; *but QED fixes it when we calculate the elastic cross section to order α^2* .

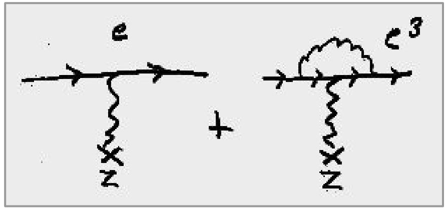
$$d\sigma_{(a)} = d\sigma_{\mathbf{elastic}} + d\sigma_{\mathbf{Brems}}(E_\gamma < \Delta E)$$

$$d\sigma_{(a)} \approx d\sigma^{(\mathbf{LO})}_{\mathbf{elastic}} \quad \leftarrow O(\alpha)$$

$$+\, d\sigma^{(\mathbf{NLO})}_{\mathbf{elastic}} + d\sigma^{(\mathbf{LO})}_{\mathbf{Brems}}(E_\gamma < \Delta E) \quad \leftarrow O(\alpha^2)$$

$$+ \text{ higher order} \quad \leftarrow O(\alpha^3)$$

- $d\sigma^{(LO)}_{\text{elastic}}$ = the Mott cross section
- $d\sigma^{(NLO)}_{\text{elastic}}$ comes from the <u>*interference*</u> of



- $d\sigma^{(LO)}_{\text{Brems}}(E_\gamma < \Delta E)$ can be approximated by soft photon production,

$$\approx d\sigma^{(LO)}_{\text{elastic}} \;\; \alpha B$$

where

αB = the soft photon emission factor

But the integral over $|\mathbf{k}|$ is infinite. We need to temporarily assume that the photon has mass λ; later, take the limit $\lambda \to 0$.

- *Analysis of B(λ)*

(We'll only do this calculation schematically.)

We have

$$\alpha B = \frac{\alpha}{(2\pi)^2}\left[\frac{2p'\cdot\epsilon}{2p'\cdot k+\lambda^2}+\frac{2p\cdot\epsilon}{-2p\cdot k+\lambda^2}\right]^2 \frac{d^3k}{E_\gamma}$$

$$\text{E.g.} \rightarrow (p'+k)^2-m^2 = 2p'\cdot k+\lambda^2$$

Or, for unpolarized Bremsstrahlung,

$$\sum_{r=1}^{3} \epsilon_r^\alpha \epsilon_r^\beta = -g^{\alpha\beta} + \frac{k^\alpha k^\beta}{\vec{k}^2}$$

does not contribute

$$B = \frac{-1}{(2\pi)^2}\int \frac{d^3k}{\sqrt{\vec{k}^2+\lambda^2}}\left[\frac{2p'^\mu}{2p'\cdot k+\lambda^2}+\frac{2p^\mu}{-2p\cdot k+\lambda^2}\right]^2 \Theta\left[\Delta E - \sqrt{\vec{k}^2+\lambda^2}\right]$$

Schematically,

$$B \sim \frac{1}{(2\pi)^2}\int_{\text{limits}} \frac{k^2 dk\, d\Omega_k}{\sqrt{\vec{k}^2+\lambda^2}} \frac{4m^2}{(2p'\cdot k+\lambda^2)^2}$$

$$= \frac{1}{(2\pi)^2}\int_\lambda^{\Delta E} \sqrt{\omega^2-\lambda^2}\,\frac{\omega\, d\omega}{\omega} \cdot \frac{4m^2\; 2\pi \sin\theta\, d\theta}{[2E\omega - 2pk\cos\theta+\lambda^2]^2}$$

approx. by $2m\omega+\lambda^2$

$$\sim \frac{4\pi}{(2\pi)^2}\int_\lambda^{\Delta E} \sqrt{\omega^2-\lambda^2}\, d\omega \frac{4m^2}{(2m\omega+\lambda^2)^2}$$

$$\underset{\lambda\to 0}{\sim} \frac{1}{\pi}\int_\lambda^{\Delta E} \sqrt{\omega^2-\lambda^2}\,\frac{d\omega}{\omega^2} = \frac{1}{\pi}\ln\left[\frac{\Delta E}{\lambda}+\sqrt{\left(\frac{\Delta E}{\lambda}\right)^2-1}\right] + \frac{1}{\pi}\frac{\lambda}{\Delta E}\sqrt{\left(\frac{\Delta E}{\lambda}\right)^2-1}$$

$$\underset{\lambda\to 0}{\sim} \frac{1}{\pi}\ln\frac{\Delta E}{\lambda} + \frac{1+\ln 2}{\pi} = \frac{1}{\pi}\ln\left(\frac{\Delta E}{\lambda}\right) + 0,5389$$

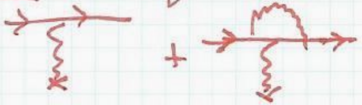
Bloch and Nordsieck:
Infrared divergences cancel out in "infrared-safe" predictions.

A complete calculation would yield

$$d\sigma^{(LO)}_{\text{soft Brems.}} \underset{\lambda\to 0}{\sim} (d\sigma)_{Mott}\ \frac{\alpha}{\pi}\left[\ln\left(\frac{\Delta E}{\lambda}\right) + C_1\right]$$

$$d\sigma^{(NLO)}_{\text{ideal elastic}} \sim (d\sigma)_{MoH}\ \frac{\alpha}{\pi}\left[-\ln\left(\frac{\Delta E}{\lambda}\right) + C_2\right]$$

actually interference of these 2 diagrams:



- soft Bremsstrahlung is divergent as $\lambda\to 0$; *(positive)*
- ideal elastic scattering interference is divergent as $\lambda\to 0$; *(negative)*
- these must be added to make a physical prediction;
- the divergences cancel!

$$d\sigma_{(a)} = (d\sigma)_{Mott}\ \frac{\alpha}{\pi}\left(C_1 + C_2\right)$$

the prediction for the experimental cross section (scattering with no observed photons) is not divergent as λ (the temporary photon mass) → 0.

We can handle IR divergences.
This is fairly easy in QED,
but quite difficult in QCD.