Chapter 9 : Radiative Corrections

9.1 Second order corrections of QED

9.2 Photon self energy

9.3 Electron self energy

9.4 External line renormalization

9.5 Vertex modification

9.6 Applications

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Chapter 10 : Regularization

10.1 Math preliminaries

10.2 Cut-off regularization

10.3 Dimensional regularization

10.4 Vacuum polarization

10.5 Anomalous magnetic moment

Ultraviolet (UV) divergences cancel after renormalization.

The parameters of the classical theory (m_o and e_0) must be "renormalized" in the quantum theory:

$$m = m_0 + \delta m$$

$$e = Z_3^{1/2} e_0$$

The renormalization constants (Z_3 and δm) are divergent integrals. These will cancel other divergent integrals.

Who proved that it works? Many people contributed. *Schwinger, Feynman, Bethe, Dyson.*

I . PREVIEW OF RENORMALIZATION

Ia. Principles of quantum field theory

We start with a field Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m_0)\psi - e_0 \bar{\psi} \not{A} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\alpha A^\alpha)^2 \qquad (\star)$$

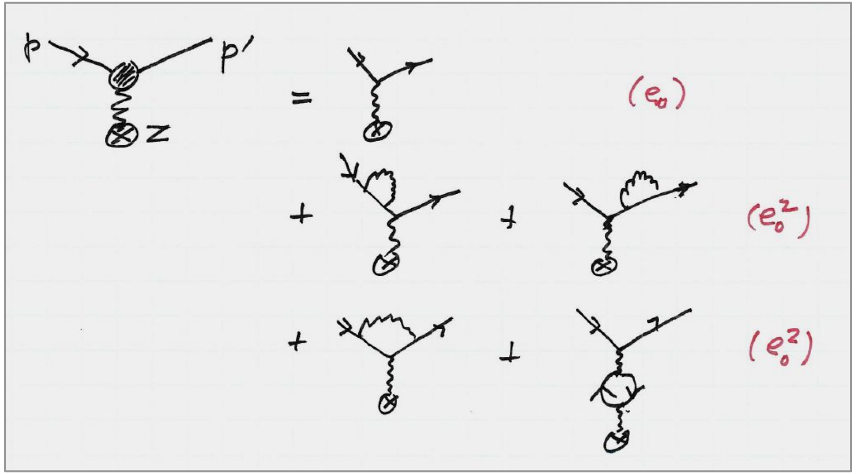
$\mathcal{L}$ depends on two theoretical parameters,

- e_0 = the bare charge (units: charge)
- m_0= the bare mass (units: mass)

Then we quantize the fields.

From $\mathcal{L}$ we can calculate transition amplitudes, using perturbation theory in e_0; i.e., Feynman diagrams.

For example, consider e-Z scattering,



There is a problem:
the loop integrals are UV divergent.

Ib . Regularization

The loop integrals are divergent, because of the contributions from virtual states with infinite momentum (UV divergent).

The electron is a point; its radius = 0; therefore the momentum transfer extends to ∞; by the uncertainty principle

So the theory (★) is not a fundamental theory, but only an *effective field theory*. We don't know what happens for infinite momentum transfer. Still, we expect that (★) will describe current experiments on electrodynamics.

Since we can't calculate with infinity, we must *regularize* the integrals – *make 'em finite*. Then when the calculation is completed, we must remove the regularization.

Methods of regularization:

- Naive momentum cutoff

Replace $\int d^3p$ by $\int d^3p \;\Theta(\Lambda - |\mathbf{p}|)$

$(\Lambda \to \infty)$

- Pauli-Villars regularization

Replace $1 /(p^2 - m^2)$ by

$$1 /(p^2 - m^2) - 1 /(p^2 - \Lambda^2)$$

$(\Lambda \to \infty)$

- Dimensional regularization
('tHooft and Veltman, 1972)

Replace $d^4p/(2\pi)^4$ by $d^Dp/(2\pi)^D$

where $D < 4$; $(D \to 4)$

Ic. Renormalizations

The theory (★) depends on two parameters, e_0 and m_0. But those are not physical observables. In particular, e_0 is not the electron charge (e), and m_0 is not the electron mass (m).

Our goal is to make predictions, e.g., cross sections, in terms of physical observables such as e and m.

So we need to "renormalize" m_0 and e_0 . That is, we must relate the "bare parameters" ($m_0$and e_0) (*which we use in [perturbation theory*) to the physical parameters (m and e).

Ultimately we'll need to introduce three renormalization constants, Z_1 , Z_2 , Z_3 ;
and, in addition, a mass renormalization constant, δm. These will be defined during the next few lectures.

$\{Z_1 , Z_2 , Z_3 , \delta m\}$ can be calculated in the regularized theory, as power series in e_0 ;
for example,

$$Z_3 = 1 + e_0^{\,2}\, \zeta_2 + e_0^{\,4}\, \zeta_4 + \ldots$$

where ζ_2 , ζ_4 , … depend on m_0 and Λ.

■ Mass renormalization

We have m_0 = "the bare mass" .
Let m = the physical electron mass.

(How is m determined?)

Now, the electron carries around some electromagnetic fields, which have energy.
We call this electromagnetic energy the electron "self-energy". Energy and mass are equivalent, so

$$\frac{m}{m_0} = 1 + \frac{\delta m}{m_0} = 1 + e_0^2 \mu_2 + e_0^4 \mu_4 + \ldots$$

The expansion coefficients μ_2 , μ_4 , μ_6 , … are calculated in the regularized theory, so they depend on m_0 and Λ (or 4 − D).

■ Charge renormalization

We have e_0 = the "bare charge".
Let e = the physical charge.

(How is e determined?)

(*There is no photon self-energy because gauge invariance requires* $m_\gamma = 0$.)

But $e \neq e_0$ because of radiative corrections.

We write $e^2 = e_0^{\,2} Z_3$ where

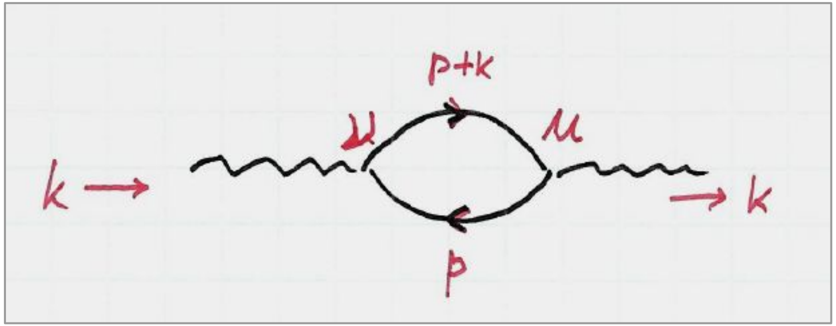
$$Z_3 = 1 + e_0^{\,2} \zeta_2 + e_0^{\,4} \zeta_4 + e_0^{\,6} \zeta_6 \ \ldots$$

The quantities $\zeta_2, \zeta_4, \zeta_6, \ldots$ are calculated in the regularized theory, so they are functions of m_0 and Λ (or $4 - D$).

Sec.9.2/ THE PHOTON SELF-ENERGY INSERTION

[The name (*photon self-energy*) is possibly misleading. <u>There is no photon mass, because of gauge invariance.</u> A better name might be *vacuum polarization insertion.***]**

Consider the lowest-order correction to the photon propagator,



IIa . Definition of the full photon propagator

The full propagator is defined *in the Heisenberg picture by*

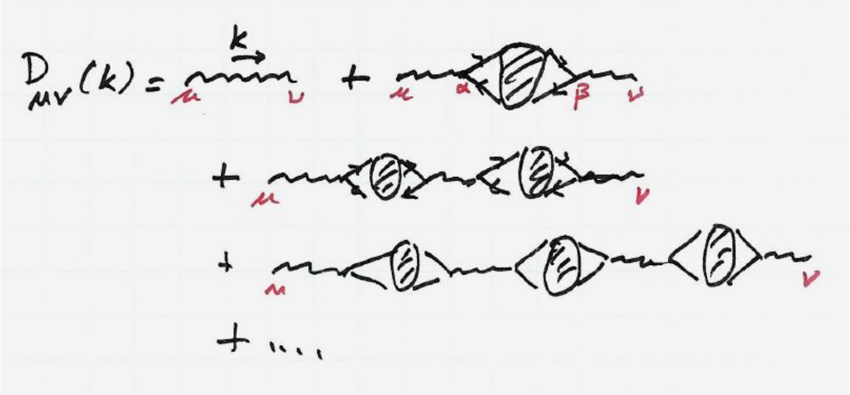
$$i D_{\mu\nu}(x-y) = \langle \Phi_0 | T A_\mu(x) A_\nu(y) | \Phi_0 \rangle$$

Transform to momentum space,

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} D_{\mu\nu}(k) e^{-ik\cdot(x-y)}$$

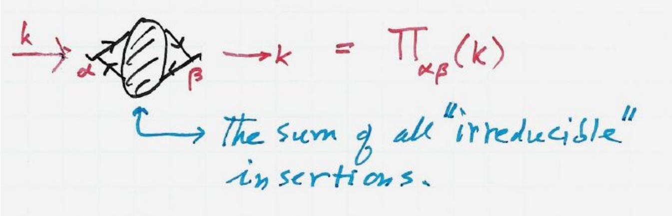
We "can" calculate $D_{\mu\nu}(q)$ in perturbation theory, using Wick's theorem;

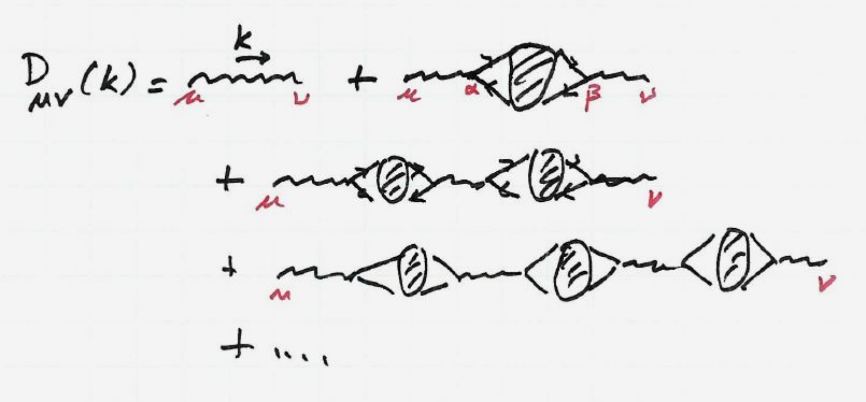
Feynman diagrams



Wherever the bare (free) photon propagator occurs in a Feynman diagram, all possible insertions must also occur, by Wick's theorem.

$\Pi_{\alpha\beta}(q)$ denotes the sum of all "irreducible" insertions.





Theorem. The ***full*** photon propagator $D_{\mu\nu}(q)$ is a geometric series in perturbation theory.

Proof ...

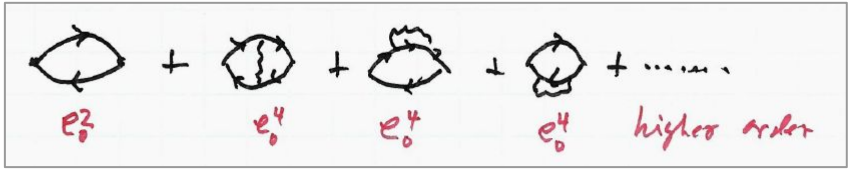
(I'm suppressing the Lorentz indices.)

$$D(k) = D_0(k) + D_0(k)\, A(k)\, D_0(k) + D_0 A D_0 A D_0 + D_0 A D_0 A D_0 A D_0 + \ldots$$

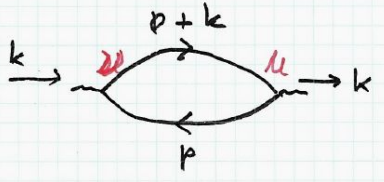
$$= D_0[1 - AD_0]^{-1} \quad ; \text{ Recall } D_0(k) = \frac{-1}{k^2}$$

$$= \frac{-1}{k^2}\left[1 + A/k^2\right]^{-1} = \frac{-1}{k^2 + A(k^2)}$$

$\Pi_{\mu\nu}(q)$ *(~ $g_{\mu\nu}\, A(k^2)$)* denotes the sum of all "irreducible" photon line insertions.



The lowest order photon line insertion, which is $O(e_0^{2})$, is



$$-e_0^2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\{ \gamma_\mu S_F(p+k) \gamma_\nu S_F(p) \}$$

$$= ie_0^{2} \, \Pi^{[2]}_{\mu\nu}(q)$$

(I have restored the Lorentz indices.)

IIb . Calculate the $O(e_0^{2})$ photon insertion

Being careful about signs and factors of i,

$$i e_0^2 \, \Pi^{(2)}_{\mu\nu}(k) = (-1)\,(ie_0)^2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\left\{ \gamma_\mu \frac{i}{\not p + \not k - m_0} \gamma_\nu \frac{i}{\not p - m_0} \right\}$$

This integral is undefined because it is UV divergent. It must be regularized.

Power counting implies a quadratic divergence
$[p^3 dp / p^2 = p \, dp]$
However, gauge invariance forces the quadratic divergence to be 0. (We'll verify that next time using dimensional regularization.)

But the integral does have a *logarithmic* UV divergence. (There is no IR divergence.)

The $O(e_0^2)$ photon insertion, *regularized*

I'll use dimensional regularization (Section 10.3). But I won't provide all the details about dimensional regularization until next time.

Replace $\int d^4p$ by $\int d^Dp$.

(Details next time)

$$e_0^2\, \Pi^{[2]}_{\mu\nu}(k) = i e_0^2 \int \frac{d^Dp}{(2\pi)^D}\, \mu^{4-D}\, \frac{\mathrm{Tr}\left\{ \gamma_\mu (\not p + \not k + m_0) \gamma_\nu (\not p + m_0) \right\}}{\left[(p+k)^2 - m_0^2 + i\epsilon\right](p^2 - m_0^2 + i\epsilon)}$$

Comment: *Why the trace? Πμν has no spinor indices; so any spinor indices must be summed. Appeal to Wick's theorem.*

What is the trace in D dimensions?

We have arrived at a regularized formula for the vacuum polarization tensor, and we'll evaluate it next time.

We'll obtain eqs (10.48) and 10.49)

$$\Pi^{[2]}_{\mu\nu}(k) = \left(k_\mu k_\nu - k^2 g_{\mu\nu}\right) \Pi^{[?]}(k^2) \qquad (10.48)$$

$$\Pi^{[?]}(k^2) = \mu^{4-D} \frac{2}{(2\pi)^{D/2}}\, \Gamma\!\left(2 - \tfrac{D}{2}\right) \int_0^1 \frac{z(1-z)\,dz}{\left[k^2 z(1-z) - m_0^2\right]^{2-D/2}} \qquad (10.49)$$

Lorentz invariance and gauge invariance

❁ $\Pi_{\mu\nu}(k)$ is a Lorentz tensor; so

$$\Pi_{\mu\nu}(k) = -\, g_{\mu\nu}\, A(k^2) + k_\mu k_\nu\, B(k^2)\ .$$

❁ $\Pi_{\mu\nu}(k)$ is gauge invariant;

so it must obey $k^\mu\, \Pi_{\mu\nu}(k) = 0.$

(footnote page 231)

Therefore,

$$-\, k_\nu\, A + k^2\, k_\nu\, B = 0$$

$$B = A\, /k^2$$

$$\Pi_{\mu\nu}(k) = \left(-\, g_{\mu\nu} + k_\mu k_\nu / k^2 \right) A(k^2)$$

❁ $D_{\mu\nu}(k)$ must have a pole at $k^2 = 0$;

so we will write $A(k^2) = k^2\, \Pi(k^2)$.

We'll obtain these results directly, using dimensional regularization, next time.)

IId. Charge renormalization and vacuum polarization to order $e_o^{\,2}$.

Look at the integral (10.49) with D = 4.

The integral over q^μ is only logarithmically divergent! By the geometric series,

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2\left[1 - \Pi(k^2)\right]} + \lambda k_\mu k_\nu$$

As usual, if we use $D_{\mu\nu}(k)$ in a calculation with Feynman diagrams, we can drop the terms $\propto\ k_\mu k_\nu$ because $k_\mu\, j^\mu(k) = 0$ by charge conservation .

Also,

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2\left[1 - e_0^2\, \Pi^{[2]}(k^2)\right]} + \underbrace{O(e_0^4)}_{\text{neglect}}$$

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2\left[1 - e_0^2\,\Pi^{[2]}(k^2)\right]}$$

Charge renormalization:

The propagator will always appear in the form

$$e_0\, D_{\mu\nu}(k)\, e_0 \equiv X_{\mu\nu}\ .$$

Now, the following expressions are accurate to order e_0^4,

$$1 - e_0^2\,\Pi^{[2]}(k^2)$$

$$= 1 - e_0^2\,\Pi^{[2]}(0) - e_0^2\,\Pi^{[2]}(k^2) + e_0^2\,\Pi^{[2]}(0)$$

$$= 1 - e_0^2\,\Pi^{[2]}(0) - e_0^2\,\Pi_c^{(2)}(k^2)$$

$$\Pi_c^{[2]}(k^2) = \Pi^{[2]}(k^2) - \Pi^{[2]}(0)$$

"Convergent part"

$$= \left(1 - e_0^2\,\Pi^{[2]}(0)\right)\left(1 - e_0^2\,\Pi_c^{[2]}(k^2)\right) + \underbrace{O(e_0^4)}_{\text{neglect}}$$

So $$X_{\mu\nu} = \frac{e_0^2(-g_{\mu\nu})}{k^2\left[1 - e_0^2\,\Pi^{[2]}(0)\right]\left[1 - e_0^2\,\Pi_c^{[2]}(k^2)\right]}$$

$$X_{\mu\nu} = \frac{e^2(-g_{\mu\nu})}{k^2\left[1 - e_0^2\,\Pi_c^{[2]}(k^2)\right]}$$

CHARGE RENORMALIZATION

$$e^2 = e_0^2\, Z_3 \quad \text{where} \quad Z_3 = \frac{1}{1 - e_0^2\,\Pi^{[2]}(0)}$$

$$\text{or} \quad Z_3 = 1 + e_0^2\,\Pi^{(2)}(0)$$

VACUUM POLARIZATION function

$$D_{\mu\nu}^{(R)}(k) = \frac{(-g_{\mu\nu})}{k^2\left[1 - e^2\,\Pi_c^{[2]}(k^2)\right]}$$

$$\text{or} \quad \frac{-g_{\mu\nu}}{k^2}\left[1 + e^2\,\Pi_c^{[2]}(k^2)\right]$$

Summary

(1) Charge renormalization, accurate to order e_0^2 ...

$$e^2 = e_0^{\,2}\, Z_3^{[2]} \qquad ;$$

with

$$Z_3^{[2]} = 1 + e_0^{\,2}\, \Pi^{[2]}(k^2{=}0)\ .$$

(2) The finite effect of vacuum polarization on the photon propagator is*

**i.e., finite as $\Lambda \to \infty$ or as $D \to 4$; the convergent part of vacuum polarization*

$$X^{[2]}_{\mu\nu} =$$

$$-g_{\mu\nu}\,(e^2/k^2)\,[\,1 + e^2\, \Pi_c^{[2]}(k^2)\,]$$

Born appx × [1 + e^2 rad.correction]

... To Be Continued .