Chapter 9 : Radiative Corrections

9.1 Second order corrections of QED

9.2 Photon self energy

9.3 Electron self energy

9.4 External line renormalization

9.5 Vertex modification

9.6 Applications

9.7 Infrared divergence

9.8 Higher order radiative corrections

9.9 Renormalizability

Chapter 10 : Regularization

10.1 Math preliminaries

10.2 Cut-off regularization

10.3 Dimensional regularization

10.4 Vacuum polarization

10.5 Anomalous magnetic moment

Some integral identities

- Euclidean momentum space,

$$\int \frac{d^4K}{(K^2+s)^2} \frac{\Lambda^2}{(K^2+\Lambda^2)}$$

$$= \pi^2 \{ \Lambda^4 /(\Lambda^2-s)^2 \ln [\Lambda^2/s] - \Lambda^2/(\Lambda^2-s)\}$$

- Spacetime with dimension D,

$$\int \frac{d^{\mathbf{D}}k}{(k^{\mathbf{2}} - s + i\,\varepsilon)^n}$$

$$= i\,\pi^{\mathbf{D/2}} (-1)^{\mathbf{n}}\, \Gamma(n-D/2)/\Gamma(n)\; s^{(\mathbf{D/2-n})}$$

The gamma function, $\Gamma(z)$;

In the limit $z \rightarrow 0$,

$$\Gamma(z) \sim 1/z - \gamma + O(z)$$

Euler constant $\gamma = 0.5772\ldots$

Pauli-Villars regularization

Suppose we encounter this integral,

$$I = \int f(k)\ d^4k \ \frac{1}{k^2 - m^2 + i\varepsilon}$$

where the factor $1/(k^2 - m^2 + i\varepsilon)$ comes from some propagator.

And the problem is that the integrand does not decrease fast enough as k increases, giving a divergent integral like $\int^\infty dk/k$,

which is UV divergent (*logarithmically*).

Replace $$\frac{1}{k^2 - m^2}$$

by $$\frac{1}{k^2 - m^2} - \frac{1}{k^2 - \Lambda^2} \quad ☆$$

➔ For finite Λ, the integral is finite. *(That's regularization.)*

➔ At the end of the calculation we will take the limit $\Lambda \to \infty$. (*For $\Lambda \to \infty$, the replacement ☆ reverts to the original integrand.)*

➔ Of course I will diverge as $\Lambda \to \infty$; but hopefully the divergence will cancel some other divergence due to renormalization.

Dimensional regularization

('tHooft and Veltman, 1972)

Again,

$$\boldsymbol{I} = \int f(k)\ d^4k \quad \frac{1}{k^2 - m^2 + i\varepsilon}$$

and the 4D integral is undefined because it is UV divergent.

Change d^4k into d^Dk where $D < 4$.

That is the regularized theory!

So, do the calculations in the regularized theory (i.e., reduced dimensions);

and at the end, take the limit $D \to 4$.

Write $D = 4 - \eta$, and finally let $\eta \to 0$.

It's a brilliant idea, if you can figure out how to do an integral over a non-integer number of dimensions.

For example, what does it mean to say $D = 3.9$? Or, what is $D = 3.9 + 0.1\, \boldsymbol{i}$?

*Here is the **definition** of integration over D dimensions:*

$$\int \frac{d^Dk}{(k^2 - s + i\,\varepsilon)^n}$$

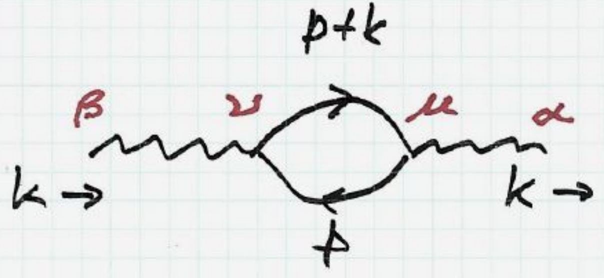
$$= i\, \pi^{D/2}\, (-1)^n\, \Gamma(n-D/2)/\Gamma(n)\; s^{(D/2-n)}$$

and obvious generalizations.

The R.H.S. is defined for all D (except D = 2n); it's sort of like analytic continuation ....

Review

The O($e_0^{\,2}$) photon self-energy insertion* is



$$= D_0^{\alpha\mu}(k)\, e_0^2\, \Pi^{[2]}_{\mu\nu}(k)\, D_0^{\nu\beta}(k)$$

* better terminology :
"vacuum polarization insertion"

$$\Pi^{[2]}_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu})\,\Pi^{[2]}(k^2)$$

The full propagator, accurate to $O(e_0^2)$

$$D_{\mu\nu}(k) = \frac{-i g_{\mu\nu}}{k^2[1 - e_0^2\Pi^{[2]}(k^2)]}$$

$$X_{\mu\nu} = e_0 D_{\mu\nu} e_0 = \frac{e_0^2(-ig_{\mu\nu})}{[1-e_0^2\Pi^{[2]}(0)]\,k^2\,[1-e_0^2\Pi_c^{[2]}(k^2)]}$$

$$= \frac{e^2(-ig_{\mu\nu})}{k^2[1-e^2\Pi_c^{[2]}(k^2)]} \quad \text{or} \quad e^2\,\frac{-ig_{\mu\nu}}{k^2}\,[1+e^2\Pi_c^{[2]}(k^2)]$$

CHARGE RENORMALIZATION

$$e^2 = \frac{e_0^2}{[1-e_0^2\Pi^{[2]}(0)]} \quad \text{or} \quad e_0^2\,[1+e_0^2\Pi^{[2]}(0)]$$

VACUUM POLARIZATION EFFECT

$$\frac{e^2}{k^2} \to \frac{e^2}{k^2}\,[1+e^2\Pi_c^{[2]}(k^2)]$$

These are the renormalization tricks

The photon propagator is

$$D = D_0 + D_0 \underbrace{e_0^2 \Pi_2}_{\hat{\Pi}_2(k^2)} D_0 + O(e_0^4)$$

neglect higher order

$$D = D_0 \left\{ 1 + e_0^2 \hat{\Pi}_2(k^2) \right\}$$

$$= \frac{D_0}{1 - e_0^2 \hat{\Pi}_2(k^2)}$$

geometric series; neglect $O(e_0^4)$

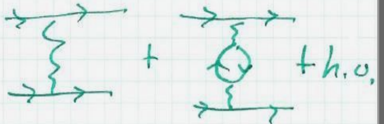
$$\hat{\Pi}_2(k^2) = \hat{\Pi}_2(0) + \underbrace{\hat{\Pi}_2(k^2) - \hat{\Pi}_2(0)}_{\Pi_{2c}(k^2)}$$

$$\text{denominator} = 1 - e_0^2 \hat{\Pi}_2(0) - e_0^2 \Pi_{2c}(k^2)$$

$$= \left[1 - e_0^2 \hat{\Pi}_2(0)\right]\left[1 - e_0^2 \Pi_{2c}(k^2)\right]$$

neglect $O(e_0^4)$

In a transition matrix element,



$$\mathcal{X} = e_0 D e_0$$

$$= \frac{e_0^2 D_0}{\left[1 - e_0^2 \hat{\Pi}_2(0)\right]} \frac{1}{\left[1 - e_0^2 \Pi_{2c}(k^2)\right]}$$

$$= \frac{e^2 D_0}{1 - e^2 \Pi_{2c}(k^2)}$$

neglect $O(e_0^6)$

- CHARGE RENORMALIZATION

$$e^2 = Z_3 e_0^2 = \frac{e_0^2}{1 - e_0^2 \hat{\Pi}_2(0)} = e_0^2 \left[1 + e_0^2 \hat{\Pi}_2(0)\right]$$

neglect $O(e_0^6)$

- VACUUM POLARIZATION

$$\mathcal{X} = e^2 D_0 \left[1 + e^2 \Pi_{2c}(k^2)\right]$$

neglect $O(e_0^6)$

Calculate the vacuum polarization tensor, ***regularized***

$$i e_0^2 \Pi^{[2]}_{\mu\nu}(k) = -e_0^2 \int \frac{d^D p}{(2\pi)^D} \mu^{4-D} \frac{N_{\mu\nu}(p,k)}{[(p+k)^2 - m_0^2 + i\epsilon](p^2 - m_0^2 + i\epsilon)}$$

where

$$N_{\mu\nu}(p,k) = \mathrm{Tr}\{\gamma_\mu(\not{p}+\not{k}+m_0)\gamma_\nu(\not{p}+m_0)\}$$

$$= f(D)\{(p+k)_\mu p_\nu + (p+k)_\nu p_\mu + [-p\cdot(p+k) + m_0^2]\, g_{\mu\nu}$$

use

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[b+(a-b)z]^2}$$ to combine the denominators

$$[\, p^2 - m_0^2 + \{[(p+k)^2 - m_0^2] - (p^2 - m_0^2)\}z \,]$$ ← to be SQUARED

$$= [\, p^2 + 2p\cdot k z + k^2 z - m_0^2 \,]$$

$$= (p + zk)^2 - k^2 z(z-1) - m_0^2$$

$$= q^2 - \underset{+i\epsilon}{\Delta^2} \quad \begin{cases} q^\mu = p^\mu + z k^\mu \\ \Delta^2 = k^2(z)(z-1) + m_0^2 \\ \quad = m_0^2 - z(1-z)k^2 \end{cases}$$

Change the variable of integration from p^μ to $q^\mu = p^\mu + z k^\mu$; $d^D p = d^D q$

$$\Pi^{[2]}_{\mu\nu}(k) = i \frac{\mu^{4-D}}{(2\pi)^D} \int_0^1 dz \int d^D q \frac{N_{\mu\nu}(q - zk, k)}{(q^2 - \Delta^2 + i\epsilon)^2}$$

$\int d^D q \; q^\mu f(q^2) = 0$ so drop the terms in $N_{\mu\nu}$ linear in q^μ ;

$$\Pi^{[z]}_{\mu\nu}(k) = i\frac{\mu^{4-D}}{(2\pi)^D}\int_0^1 dz \int d^D q \, \frac{N_{\mu\nu}(q-zk,k)}{(q^2-\Delta^2+i\epsilon)^2}$$

$$N_{\mu\nu} = f(D)\{ [q+(1-z)k]_\mu [q-zk]_\nu + [q+(1-z)k]_\nu [q-zk]_\mu + [(q-zk)\cdot(q+(1-z)k)+m_0^2]\, g_{\mu\nu} \}$$

$$\{\ldots\} = \begin{array}{ll} 2q_\mu q_\nu - q^2 g_{\mu\nu} & I_1 \\ + k_\mu k_\nu[-2z(1-z)] + z(1-z)k^2 g_{\mu\nu} + z(1-z)k^2 g_{\mu\nu} & \Big\} I_3 \\ + m_0^2 g_{\mu\nu} - z(1-z)k^2 g_{\mu\nu} & I_2 \end{array}$$

(neglect linear)

let $J_n(s) = \int d^D q \, \frac{1}{[q^2-s+i\epsilon]^n} = i\pi^{D/2}(-1)^n \frac{\Gamma(n-D/2)}{\Gamma(n)\, s^{n-D/2}}$

derivation at end

$$I_1 = \int d^D q \, \frac{2q_\mu q_\nu - q^2 g_{\mu\nu}}{[q^2-\Delta^2+i\epsilon]^2}$$

Tricks

- $q_\mu q_\nu = g_{\mu\nu} q^2/D$
- $q^2 = q^2 - \Delta^2 + \Delta^2$

$$I_1 = \left(\frac{2}{D} g_{\mu\nu} - g_{\mu\nu}\right)\left(J_1(\Delta^2) + \Delta^2 J_2(\Delta^2)\right)$$

$$I_2 = [m_0^2 - z(z-1)k^2] g_{\mu\nu} \; J_2(\Delta^2) = \Delta^2 J_2(\Delta^2)$$

$$I_1 + I_2 = g_{\mu\nu}\left\{\left(\frac{2}{D}-1\right)(J_1 + \Delta^2 J_2) + \Delta^2 J_2\right\} = 0$$

derivation at end

Put everything together …

$$\Pi^{[2]}_{\mu\nu}(k) = i\,\frac{\mu^{4-D}}{(2\pi)^D}\int_0^1 dz\, f(D)\, I_3$$

$$I_3 = \int d^Dq\, \frac{2z(1-z)(k^2 g_{\mu\nu} - k_\mu k_\nu)}{(q^2-\Delta^2+i\epsilon)^2}$$

$$= 2z(1-z)(k^2 g_{\mu\nu} - k_\mu k_\nu)\, J_2(\Delta^2)$$

$$J_2(\Delta^2) = 2i\pi^{D/2}\,\frac{\Gamma(2-D/2)}{\Gamma(2)}\,(\Delta^2)^{D/2-2}$$

$$\Pi^{[2]}_{\mu\nu}(k) = (k_\mu k_\nu - k^2 g_{\mu\nu})\,\mu^{4-D}\,\frac{f(D)}{(4\pi)^{D/2}}\,\Gamma(2-D/2)\int_0^1 2z(1-z)\,dz\,(\Delta^2)^{D/2-2}$$

where $\Delta^2 = m_0^2 - z(1-z)k^2$

$$= (k_\mu k_\nu - k^2 g_{\mu\nu})\,\Pi^{[2]}(k^2)$$

Now take the limit D $\rightarrow$ 4.
Let D = 4 − η and η $\rightarrow$ 0 .

$$\Pi^{[2]}(k^2) = \mu^\eta\,\frac{f(4-\eta)}{(4\pi)^{2-\eta/2}}\,\Gamma\!\left(\frac{\eta}{2}\right)\int_0^1 (\Delta^2)^{-\eta/2}\,2z(1-z)\,dz$$

Asymptotic expansions
As x $\rightarrow$ 0,

$$\Gamma(x) \sim 1/x - \gamma + (x/12)(6\gamma^2 + \pi^2) + \ldots$$
$$A^x \sim 1 + x \ln A + \tfrac{1}{2} x^2 (\ln A)^2 + \ldots$$
$$\Gamma(\eta/2) \sim 2/\eta - \gamma$$

the UV divergence!

Calculate the limit carefully…

Euler constant $\gamma =$
$\lim \{1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + 1/N - \ln(N)\} = 0.5772$

Now take the limit $\eta = 4 - D \to 0$.

$$F(\eta) = \mu^{\eta}\ f(4-\eta) / (4\pi)^{2-\eta/2} \int (\Delta^2)^{-\eta/2}\, 2z(1-z)dz$$

$$\Pi^{[2]}(k^2) = \Gamma(\eta/2)\, F(\eta)$$

$$\sim (2/\eta - \gamma)\, (F(0) + \eta F'(0))$$

$$\sim (2/\eta) F(0) - \gamma F(0) + 2 F'(0) + \mathcal{O}(\eta)$$

divergent but constant; convergent

$$F(0) = f(4) / (4\pi)^2 \int_0^1 2z(1-z)\, dz$$

$$= 4/(4\pi)^2\ 2(1 - ½) = 1/(4\pi^2)$$

$$F'(0) = (dF/d\eta)\,\big|_{\eta=0}$$

= carefully ...

And $F'(0)$ =

$$\ln(\mu)\, F(0) - f'(4)\, F(0)/4 + ½ \ln(4\pi)\, F(0)$$

$$+ f(4)/(4\pi)^2 \int_0^1 [-½ \ln(\Delta^2)]\, 2z(1-z)\, dz$$

$1/(4\pi^2)$ $\Delta^2 = m_0^2 - z(1-z)\, k^2$

We'll write it this way:

$$\Pi^{[2]}(k^2) = \Pi^{[2]}(k^2{=}0) + \Pi_C^{[2]}(k^2)$$

$$\Pi^{[2]}(k^2{=}0) = \{ 2/\eta - \gamma - 2f'(4)/4 + \ln(4\pi) + \ln(\mu^2/m_0^2) \} \times (1/4\pi^2)$$

Charge renormalization to $O(e_0^2)$ from the vacuum polarization insertion to the photon propagator.

$$e^2 = Z_3\, e_0^2$$

$$Z_3 = 1 + e_0^2\, \Pi^{[2]}(k^2{=}0) \quad + O(e_0^4)$$

$$Z_3 = 1 + e_0^2 \left(\frac{1}{2\pi^2} \frac{1}{\eta} + C\mathit{onstant} \right)$$

where $\eta = 4 - D \to 0$.

Or, for Pauli Villars regularization,

replace $1/\eta$ by $\mathit{ln}\,(\Lambda^2 / m_0^2)$

where $\Lambda \to \infty$.

Naively,
we could say $Z_3 = \infty$;
which would require $e_0 = 0$.

But in reality, QED is an "effective field theory", valid for energies below the large cutoff Λ (? ~ m_{Planck}
~ 10^{19} GeV ?).

We do not know the theory for higher energies.

So *in reality* $e_0^2 \ln(\Lambda^2 / m_0^2)$ is a small number; even though we take the limit $\Lambda \to \infty$ when we calculate the convergent part.

Why we tolerate "infinite renormalization"

Consider:

Let Λ = the Planck mass,

$$\Lambda = (\hbar c/G_{Newton})^{1/2} = 2.2 \times 10^{-8} \text{ kg}$$

$$m_e = 9.1 \times 10^{-31} \text{ kg}$$

$$Z_3^{[2]} \sim \mathbf{1} + \alpha\,(\log(\Lambda/m) + C)$$

$$= 1 + 1/137 \times \ln(0.2 \times 10^{23})$$

$$= 1 + 0.37$$

Viewing QED as an effective theory,

the charge renormalization is not large.

Paul Dirac quote:

"Renormalization is just a stop-gap procedure. There must be some fundamental change in our ideas, probably a change just as fundamental as the passage from Bohr's orbit theory to quantum mechanics. When you get a number turning out to be infinite which ought to be finite, you should admit that there is something wrong with your equations, and not hope that you can get a good theory just by doctoring up that number."

Any ideas?

- Physics beyond the S. M.
  (probably not radical enough)
- Quantum gravity
- String theory, 10 dimensions, etc.

The physical consequences of the ***convergent part*** *of the vacuum polarization diagram*

$$\Pi_C^{[2]}(k^2) = \Pi^{[2]}(k^2) - \Pi^{[2]}(k^2{=}0)$$

$$\Pi_C^{[2]}(k^2) = 2\,(1/4\pi^2)\ \int_0^1 [-\tfrac{1}{2}\,ln(\Delta^2/m_0^2)]\ 2z(1\text{-}z)\ dz$$

where

$$\Delta^2/m_0^2 = 1 - z(1-z)\ k^2/m_0^2\ .$$

That is,

$$e^2\Pi_C^{[2]}(k^2) = -\,(2\alpha/\pi)\ \int_0^1 dz\ z\,(1-z)\ ln\,(1 - k^2\,z\,(1{-}z)\,/m_0^2) \qquad (9.68)$$

Read Section 9.6.2

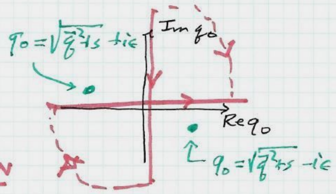
➔ the Lamb shift (1947);
The Dirac equation has $E(2s_{1/2}) = E(2p_{1/2})$.
The Lamb shift is $\Delta E/\hbar$. The vacuum polarization contributes 27 MHz to the Lamb shift; the total Lamb shift is 1028 MHz.

➔ "vacuum polarization"
The QED interaction is stronger at short distances because polarization of the virtual electron-positron pairs screens the charges.

➔ But ***in QCD,*** the interaction is weaker at short distances, which is called *"asymptotic freedom"*. Gluon fluctuations anti-screen the color charges.

Derivations

$$J_n(s)=\int \frac{d^Dq}{(q^2-s+i\epsilon)^n} = \int \frac{d^{D-1}q\,dq_0}{(q_0^2-\vec{q}^{\,2}-s+i\epsilon)^n}$$



WICK ROTATION

$$= -\int_{i\infty}^{-i\infty} \frac{d^{D-1}q\,dq_0}{(q_0^2-\vec{q}^{\,2}-s)^n} \quad ;\text{let } q_0=-iQ$$

$$= -\int_{-\infty}^{\infty}\frac{d^{D-1}q\,(-i)\,dQ}{(-Q^2-\vec{q}^{\,2}-s)^n} = i\int \frac{d^DQ}{(Q^2+s)^n}(-1)^n$$

in Euclidean D space (Q^μ)

$$= i(-1)^n\int_0^\infty \frac{Q^{D-1}dQ}{(Q^2+s)^n}\;\frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Surface area of a unit sphere in D dimensions

$$= i(-1)^n \frac{2\pi^{D/2}}{\Gamma(D/2)}\, s^{D/2-n}\underbrace{\int_0^\infty\frac{x^{D-1}dx}{(x^2+1)^n}}_{\frac{\Gamma(D/2)\,\Gamma(n-D/2)}{2\Gamma(n)}}$$

$$= i(-1)^n\pi^{D/2}\frac{\Gamma(n-D/2)}{\Gamma(n)}\,s^{D/2-n}$$

THIS IS THE DEFINITION OF INTEGRATION in D DIMENSIONS

$$J_n(s)=\int d^Dq\,\frac{1}{[q^2-s+i\epsilon]^n} = i\pi^{D/2}(-1)^n\,\frac{\Gamma(n-D/2)}{\Gamma(n)\,s^{n-D/2}}$$

$$\left(\frac{2}{D}-1\right)\left(J_1(\Delta^2)+\Delta^2 J_2(\Delta^2)\right) + \Delta^2 J_2$$

$$= \left(\frac{2}{D}-1\right) i\pi^2(-1)^1\,\frac{\Gamma(1-D/2)}{\Gamma(1)\,\Delta^{2(1-D/2)}} + \frac{2}{D}\Delta^2\, i\pi^2(-1)^2\,\frac{\Gamma(2-D/2)}{\Gamma(2)\,(\Delta^2)^{2-D/2}}$$

$$= i\pi^2\,(\Delta^2)^{D/2-1}\,\Gamma(1-D/2)$$

$$\left\{-\left(\frac{2}{D}-1\right)+\frac{2}{D}\left(1-\frac{D}{2}\right)\right\}$$

$$=0$$

$\Gamma(1)=0!=1$

$\Gamma(2)=1!=1$

$\Gamma(n+1)=n\,\Gamma(n)$

Homework Problems

due Friday April 28

30. Mandl and Shaw problem 9.2.

31. Mandl and Shaw problem 10.2.

32. Mandl and Shaw problem 10.3.

33. For the electron, ***e*** and ***m*** are known to high accuracy. Explain how ***e*** and ***m*** are measured.

34. Show that the QED interaction is stronger at short distances (or, large momentum transfer) due to vacuum polarization.