PHY410 Homework Set 1

1. [5 pts] Read carefully the introductory section of Chapter 1 in the textbook. Consider a particle of mass m confined in a 2-dimensional square box of size $L\times L$. The potential energy of the particle has the form

$$U(x,y) = \begin{cases} 0\,, & \text{if } 0 < x,\ y < L, \\ \infty\,, & \text{otherwise.} \end{cases}$$

Find the multiplicity (degeneracy) of the 8 lowest energy levels.

2. [5 pts] The exact and approximate expressions for the multiplicity of N spins with spin excess $2s$ are respectively

$$g(N,s) = \frac{N!}{(N/2+s)!\,(N/2-s)!}\,, \quad \text{and} \quad g(N,s) \approx g(N,0)\,\mathrm{e}^{-2s^2/N}\,.$$

Make a plot within any software you find convenient to use, including possibly the on-line Wolfram Alpha (http://www.wolframalpha.com), of either $g(100,s)$ or of $\ln g(100,s)$, for the two expressions above. Employ the range $-18 < s < 18$ and overlay the two plots. Comment on the results. Note: In class we found $g(N,0) \approx 2^N \sqrt{\frac{2}{\pi N}}$. However, for the sake of this problem use the exact expression $g(N,0) = \frac{N!}{[(N/2)!]^2}$.

3. [10 pts] The saddle point approximation pertains to situations where an integral of the form

$$I = \int_{-\infty}^{\infty} \mathrm{d}x\, g(x)\,\mathrm{e}^{-f(x)}\,,$$

is evaluated and f has a narrow minimum at some x_0. Around the minimum, f can be approximated as

$$f(x) \approx f(x_0) + \frac{1}{2}\,(x-x_0)^2\, f\text{"}(x_0)\,,$$

and the integral above can be then approximated as

$$I \approx \mathrm{e}^{-f(x_0)} \int_{-\infty}^{\infty} \mathrm{d}x\, g(x)\,\mathrm{e}^{-\frac{1}{2}\,(x-x_0)^2\, f\text{"}(x_0)}\,.$$

For a narrow minimum in f, g can be further expanded around x_0, with only $g(x_0)$ contributing in the leading order to the integral. With this, one finds under the saddle-point approximation

$$I =\approx g(x_0)\,\mathrm{e}^{-f(x_0)} \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{e}^{-\frac{1}{2}\,(x-x_0)^2\, f\text{"}(x_0)} = g(x_0)\,\mathrm{e}^{-f(x_0)} \sqrt{\frac{2\pi}{f\text{"}(x_0)}}\,.$$

In class, we implicitly employed the saddle-point approximation for the spin system and the validity of the approximation in that case was ensured by having a large number of spin sites N. Next-order contributions stem from including further terms of expansion both for g and f under the integral.

Consider next the integral

$$I = \int_0^\infty \mathrm{d}x \, x \, \mathrm{e}^{-ax-\frac{b}{\sqrt{x}}} \, ,$$

where a, $b > 0$. Do as much as you can from the tasks below.

(a) Find the location of the saddle point x_0 in terms of a and b.

(b) By examining the second and possibly higher derivatives of the argument of the exponential at the saddle point, and by considering the x-factor multiplying the exponential, determine the conditions on a and b that should be met in order to make the saddle-point approximation a good approximation.

(c) Evaluate the integral above employing the saddle point approximation, while using $x \approx x_0$ for the x-factor multiplying the exponential. Would using $x = x_0 + (x - x_0)$ produce a different result in the saddle-point approximation?

(d) For $a = 1$ and $b = 5$, compute I numerically using a calculator or the Wolfram Alpha site (http://www.wolframalpha.com). If the upper limit of infinity cannot be used, take a large value for the upper limit, such as $x_{\max} = 20$. Move the upper limit up and down, to make sure that the numerical integral has converged. Compare the result from the saddle-point approximation above to the numerical result. What error in percentage is made using the saddle-point approximation? Note: Do not expect any high accuracy here.

(e) Discuss how the accuracy of the saddle-point approximation could be improved for the specific calculation above by inclusion of higher order terms under the integral. Indicate what those terms would be. You do not need to carry out an explicit calculation for those terms.