PHY 855 Assignment #1

Problem 1

Ground state $\Phi_0(x) = \langle x|0\rangle$

where $a|0\rangle = 0$ and $a = \sqrt{\frac{m\omega}{2\hbar}}\left(q + \frac{ip}{m\omega}\right)$.

Now, $\langle x|a|0\rangle = 0 = \sqrt{\frac{m\omega}{2\hbar}}\langle x|q + \frac{ip}{m\omega}|0\rangle$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left\{x\,\Phi_0(x) + \frac{i}{m\omega}\frac{\hbar}{i}\frac{\partial}{\partial x}\Phi_0(x)\right\}$$

So, $$\frac{\partial \Phi_0}{\partial x} = -\frac{m\omega}{\hbar}\,x\,\Phi_0$$

$$\therefore \quad \Phi_0(x) = N\,e^{-\frac{1}{2}\alpha x^2} \qquad (N:\text{ normalization})$$

where $-\alpha x = -\frac{m\omega}{\hbar}x$; $\alpha = \frac{m\omega}{\hbar}$.

Result, $$\boxed{\Phi_0(x) = N\,e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2}.}$$ 2 points

Problem 2 Consider the state $|\psi\rangle = \sum_n c_n |n\rangle e^{-iE_n t/\hbar}$

where $c_n = e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{n!}}$ (coherent state).

(A) Start with the hint: $a|t\rangle = ?$

$$= \sum_n c_n \underbrace{a|n\rangle}_{\sqrt{n}|n-1\rangle} e^{-iE_n t/\hbar} \qquad (\text{set } \hbar = 1)$$

$$= \sum_n e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle e^{-i(E_{n-1}+\hbar\omega)t}$$

$$= \alpha e^{-i\omega t} \sum_n e^{-\alpha^2/2} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle e^{-iE_{n-1}t}$$

$$= \alpha e^{-i\omega t} |t\rangle$$

$$\begin{cases} a|t\rangle = \alpha e^{-i\omega t} |t\rangle \\ \langle t| a^+ = \alpha^* e^{i\omega t} \langle t| \end{cases}$$

$\alpha^* = \alpha$; assume α is real.

Now $x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+)$

so $$\langle t| x |t\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\underbrace{\langle t|a|t\rangle}_{\alpha e^{-i\omega t}} + \underbrace{\langle t|a^+|t\rangle}_{\alpha e^{i\omega t}} \right)$$

$$\boxed{\langle t|x|t\rangle = \sqrt{\frac{2\hbar}{m\omega}}\, \alpha \cos\omega t = A \cos\omega t \quad \text{and} \quad A = \sqrt{\frac{2\hbar}{m\omega}}\, \alpha .}$$

2 points

(B) $\langle t|x^2|t\rangle = \frac{\hbar}{2m\omega} \langle t| a^2 + (a^+)^2 + aa^+ + a^+a |t\rangle$

$a^2 \to \alpha^2 e^{-2i\omega t}$; $(a^+)^2 \to \alpha^2 e^{+2i\omega t}$;

$a^+a \to \alpha^2$ and $aa^+ = 1 + a^+a \to 1 + \alpha^2$

$\Downarrow$

$$\langle t|x^2|t\rangle = \frac{\hbar}{2m\omega} \alpha^2 \left[2\cos(2\omega t) + 2 + \frac{1}{\alpha^2} \right]$$

$$= \frac{\hbar}{2m\omega} + \frac{\hbar}{2m\omega}\alpha^2 . 2 . 2\cos^2(\omega t)$$

$$= \frac{\hbar}{2m\omega} + \frac{2\hbar}{m\omega}\alpha^2 \cos^2\omega t = \frac{\hbar}{2m\omega} + A^2\cos^2\omega t$$

The uncertainty of x is Δx, where

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} + A^2\cos^2\omega t - A^2\cos^2\omega t$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}$$

Compare $\boxed{\frac{\Delta x}{A} = \frac{\sqrt{\hbar/2m\omega}}{\sqrt{2\hbar/m\omega}\,\alpha} = \frac{1}{2\alpha}}$ 2 points

In the classical limit, α is large and $\frac{\Delta x}{A}$ is small.

(C) $\langle t|H|t\rangle = \hbar\omega \langle t| a^+a + \frac{1}{2} |t\rangle$

$$= \hbar\omega \left(\alpha^2 + \tfrac{1}{2}\right)$$

The classical energy is $E_{classical} = \frac{1}{2}m\omega^2 A^2 = \hbar\omega\alpha^2$;

So $\boxed{\langle t|H|t\rangle = E_{classical} + \text{zero point energy.}}$ 2 points

Problem 3 Given

$$\vec{A}(\vec{x},t) = \sum N_k \vec{\epsilon}_{\vec{k}\sigma} \{ a_{\vec{k}\sigma} e^{i(\vec{k}\cdot\vec{x}-\omega t)} + a^+_{\vec{k}\sigma} e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \}$$

and $H = \sum \hbar\omega \, a^+_{\vec{k}\sigma} a_{\vec{k}\sigma} + \text{const.}$

(a) To prove that $\vec{A}(\vec{x},t)$ obeys the Heisenberg equation of motion,

$$\boxed{-i\hbar \frac{\partial \vec{A}}{\partial t} = [H, \vec{A}].}$$

2 points

- Left hand side $= -i\hbar \sum N\vec{\epsilon} \{ a\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} (-i\omega) + a^+ e^{-i(\vec{k}\cdot\vec{x}-\omega t)} (i\omega) \}$

$$= \sum N\vec{\epsilon} \{ -\hbar\omega \, a\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \hbar\omega \, a^+ e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \}$$

- Right hand side $= \sum N\vec{\epsilon} \{ \hbar\omega [a^+a, a] e^{i(\vec{k}\cdot\vec{x}-\omega t)} + \hbar\omega [a^+a, a^+] e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$

$$[a^+a, a] = a^+aa - aa^+a = [a^+, a]a = -a$$

$$[a^+a, a^+] = a^+aa^+ - a^+a^+a = a^+[a, a^+] = +a^+$$

So right hand side $= \sum N\vec{\epsilon} \{ -\hbar\omega \, a e^{i(k\cdot x-\omega t)} + \hbar\omega \, a^+ e^{-i(k\cdot x-\omega t)} \}$

which is the same as the left-hand side. Q.E.D.

Problem 3 part B

We have $\vec{A}(\vec{x},t) = \sum_{k\sigma} N_k \vec{\epsilon}_k \left\{ a_k e^{i\Phi} + a_k^+ e^{-i\Phi} \right\}$ $\qquad \Phi = \vec{k}\cdot\vec{x} - \omega t$

$\vec{E} = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t} = \sum N_k \vec{\epsilon}_k \left\{ \frac{i\omega}{c} \right\} \left\{ a_k e^{i\Phi} - a_k^+ e^{-i\Phi} \right\}$

$\vec{B} = \nabla \times \vec{A} = \sum N_k (i\vec{k} \times \vec{\epsilon}_k) \left\{ a_k e^{i\Phi} - a_k^+ e^{-i\Phi} \right\}$

Field energies:

$$U_E = \frac{1}{2}\int E^2 d^3x = \frac{1}{2}\sum_{k_1}\sum_{k_2} N_{k_1} N_{k_2} \frac{i\omega_1}{c}\frac{i\omega_2}{c} \int d^3x \sum_{\sigma_1}\sum_{\sigma_2} \vec{\epsilon}_1 \cdot \vec{\epsilon}_2$$
$$\left(a_{k_1} e^{i\Phi_1} - a_{k_1}^+ e^{-i\Phi_1}\right)\left(a_{k_2} e^{i\Phi_2} - a_{k_2}^+ e^{-i\Phi_2}\right)$$

The coordinate integral:

$$\int d^3x \; e^{\pm i\vec{k}_1\cdot\vec{x}} e^{\pm i\vec{k}_2\cdot\vec{x}} = \Omega\, \delta_{Kr}(\pm\vec{k}_1 \pm \vec{k}_2)$$ (Ω = finite volume; periodic boundary conditions)

$\therefore$

$$U_E = \frac{\Omega}{2}\sum_{k_1} 2 N_{k_1}^2 k_1^2 \left[-a_{k_1} a_{-k_1} e^{-2i\omega_1 t} - a_{k_1}^+ a_{-k_1}^+ e^{2i\omega_1 t} + a_{k_1} a_{k_1}^+ + a_{k_1}^+ a_{k_1} \right]$$

2 polarizations

Similarly, [note: $(\vec{k}\times\vec{\epsilon})\cdot(\vec{k}\times\vec{\epsilon}) = k^2$] for the magnetic energy $U_B = \int \frac{1}{2} B^2 d^3x$:

$$U_B = \frac{\Omega}{2}\sum_{k_1} 2N_{k_1}^2 k_1^2 \left[a_{k_1} a_{-k_1} e^{-2i\omega_1 t} + a_{k_1}^+ a_{-k_1}^+ e^{2i\omega t} + a_{k_1} a_{k_1}^+ + a_{k_1}^+ a_{k_1} \right]$$

$$U = U_E + U_B = \Omega \sum_k N_k^2 k^2 (a_k a_k^+ + a_k^+ a_k)$$
$$= \Omega \sum_k N_k^2 \left(\frac{\omega}{c}\right)^2 (2a_k^+ a_k + 1) = \sum_{k\sigma} \hbar\omega\left(a_k^+ a_k + \tfrac{1}{2}\right)$$

To be equivalent we must require

$$\boxed{N_k^2 = \frac{\hbar c^2}{2\omega\Omega}}$$

2 points

Problem 4

Time evolution in the Schroedinger picture is

$$|\alpha, t\rangle_S = e^{-iHt/\hbar} |\alpha, 0\rangle_S \; ;$$

time evolution in the Heisenberg picture is

$$O_H(t) = e^{iHt/\hbar} \, O_H(0) \, e^{-iHt/\hbar} .$$

Now consider the matrix elements :

$$\langle \alpha, t | O | \beta, t \rangle_{Sch.} = \langle \alpha, 0 | \underbrace{e^{iHt/\hbar} \, O \, e^{-iHt/\hbar}}_{O_H(t)} |\beta, 0\rangle$$

↑ Heisenberg state

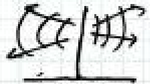
$$= \langle \alpha | \, O(t) \, | \beta \rangle_{Hei.}$$

and that proves that the matrix elements are the same in both pictures.

$\boxed{\text{Q.E.D.}}$

2 points

Problem 5



WKAR

$f = 90.5 \times 10^6\ s^{-1}$; $\omega = 2\pi f$

$P = 85 \times 10^3\ W$

(A) Let $\frac{dn}{dt}$ = mean number of photons radiated per second

Then $P = \frac{dE}{dt} = \hbar\omega \frac{dn}{dt} = 2\pi\hbar f \frac{dn}{dt}$.

The mean number of photons emitted in 1 second is

$$\delta n = \frac{dn}{dt}\delta t = \boxed{1.417 \times 10^{30}}$$

(B) $(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \langle n \rangle$ for the coherent state.

Therefore $\Delta n = \sqrt{\langle n \rangle} = \boxed{1.2 \times 10^{15}}$

4 points

Problem 6 The photon coherent state is

$$|\psi, t\rangle = \sum_{n=0}^{\infty} c_n |n\rangle e^{-iE_n t/\hbar}$$

where $c_n = e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{n!}}$.

Now consider $a|\psi, t\rangle$.

$$a|\psi\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\, |n-1\rangle e^{-iE_n t/\hbar}$$

$$= \sum_{n=1}^{\infty} e^{-\alpha^2/2} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle e^{-i(E_{n-1}+\hbar\omega)t/\hbar}$$

$$= \alpha e^{-i\omega t} |\psi\rangle$$

The eigenvalue is $\boxed{\alpha e^{-i\omega t}}$.

2 points

Problem 3 part B

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