Homework Assignment #2

Problem 7

Calculate the mean lifetime of the 2p state of hydrogen.

Start with the general formula for the decay rate (Equation 1.53).

$$w_{total}(A \to B) = e^2\, \omega^3 / (3\pi\, \hbar\, c^3) \quad |\langle B|\mathbf{x}|A\rangle|^2 \;.$$

For 2p → 1s, calculate the matrix element,

$$\langle 1s|\mathbf{x}|2p,m{=}0\rangle = \mathbf{e_z} \;\; [\, 2^7\sqrt{(2)}\, /3^5\,]\; a_B \;.$$

Use these constants:

$$e^2 = 4\pi\, \hbar c\, \alpha = \; 4\pi\, \hbar c\, /137$$

$$\omega = (\, \epsilon_{2p} - \epsilon_{1s}\,)\, /\, \hbar = \tfrac{3}{4}\, Ry\, /\hbar = \tfrac{3}{4}\, (\tfrac{1}{2}\, mc^2)\, \alpha^2$$

$$a_B = 4\pi\, \hbar^2\, /\, (me^2) = \hbar\, /(mc\, \alpha)$$

Thus

$$w = (256/6561)\, (mc^2/\hbar)\, \alpha^5 = 6.27 \text{ x } 10^8 \text{ s}^{-1}$$

The mean lifetime is

$$T \;=\; 1\,/w \;=\; 1.59 \text{ x } 10^{-9} \text{ s.}$$

2 points

Problem 8

(a) Estimate the decay rate for a nuclear gamma decay .

(b) Compare the result to the "Weisskopf estimate",

$\Gamma_W = (0.068 \text{ eV}/\hbar)\, A^{2/3}\, (\Delta E/\text{MeV})^3$.

(c) Estimate the mean lifetime, for A = 40 and ΔE = 1 MeV.

(a, b) Start with the general formula for the decay rate (Equation 1.53) for an electric dipole transition,

$$w_{total}(A \rightarrow B) = e^2\, \omega^3 / (3\pi\, \hbar\, c^3) \;\; |\langle B|\mathbf{x}|A\rangle|^2 \;\; ;$$

that is for one charge e making an energy transition, and $\hbar\omega = \Delta E$.

- Treat the nucleus as a bag of particles (protons and neutrons). One proton makes a transition to a lower energy state and emits a photon; $E_1 \rightarrow E_2 + \gamma$.
- Estimate $|\langle B|\mathbf{x}|A\rangle|^2 \sim R^2$

  where R = nuclear radius = $r_0 A^{1/3}$ with r_0 = 1.25 fm.
- So we have

  decay rate = $w \sim 4\pi\hbar c\, \alpha\, (\Delta E/\hbar)^3\, (3\pi\, \hbar c^3)^{-1}\, \mathbf{r_0^2}\, A^{2/3}$

  $= A^{2/3}\, (\Delta E/\text{MeV})^3\, \mathbf{C}$ 2 points

  where **C** = $4/3\;\; \alpha\;\; [(1 \text{ MeV} \times r_0)/(\hbar c)]^2\, (\text{MeV}/\hbar)$ = **0.39 eV /ħ**
  which resembles with the Weisskopf estimate.

(c) Using the Weisskopf estimate, the mean lifetime is

$$T = \hbar/\Gamma_w = 8 \times 10^{-\mathbf{16}} \text{ sec.}$$ 2 points

Problem 9

The Thomson cross section

$$d\sigma / d\Omega = \alpha^2 (\hbar / mc)^2 (1 - ½ \sin^2 \theta)$$

where $\alpha = e^2 /(4\pi\hbar c) = 1/137$.

The *total* cross section is

$$\sigma_T = (8\pi/3) \alpha^2 (\hbar / mc)^2 = 66.5 \text{ fm}^2 = \boxed{665 \text{ mb}}$$ 2 points

(1 barn = 10^{-28} m^2)

Problem 10

The photon, created at $r = 0$, undergoes a random walk until it reaches $r = R$. Then it escapes.

Calculate the mean time to escape.

/1/ Model assumptions

- the sun has constant mass density;
  $M = 1.99 \times 10^{30}$ kg
  $R = 6.96 \times 10^{8}$ m
  $\rho = M / V = 1.408 \times 10^{3}$ kg/m^3
  n_e = electron density = $\rho / m_{proton} = 8.44 \times 10^{29}$ m^{-3}
- the photon interacts only by Thomson scattering from electrons;
  $\sigma = 6.65 \times 10^{-29}$ m^2

/2/ The mean free path ≡ λ

$n_e \, \sigma \, \lambda = 1$ so $\boxed{\lambda = 1.78 \text{ cm}}$ 2 points

/3/ The random walk; or, diffusion to the surface

- Let D = the mean distance from the origin ($r = 0$) after N random steps. Then $D^2 = N \lambda^2$; so, $D = \sqrt{(N)}\, \lambda$.
- The photon escapes when D = R, so the mean number of steps to escape is $N = R^2 / \lambda^2 = 1.53 \times 10^{21}$.
- The time for each random step is (on average)
  $\tau = \lambda / c = 1.78 \text{ cm} / (3 \times 10^{10} \text{ cm/s}) = 0.59 \times 10^{-10}$ s
- The total time to escape is

$\boxed{\text{Time} = N\tau = 9.06 \times 10^{10} \text{ s} = 2900 \text{ years.}}$ 2 points