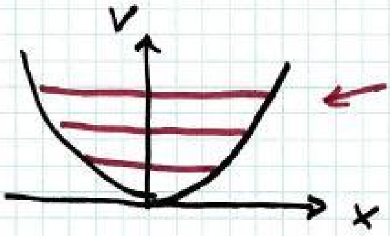
Homework Set 3

Problem 11 3 spin 0 particles in a harmonic oscillator potential



← single particle energy levels

$\epsilon_n = \hbar\omega(n + 1/2)$

$u_n(x)$

The three particle energy $= E = \hbar\omega(n_1 + \frac{1}{2} + n_2 + \frac{1}{2} + n_3 + \frac{1}{2})$

$= \hbar\omega(n_1 + n_2 + n_3 + 3/2) = \frac{9}{2}\hbar\omega.$

Therefore, $n_1 + n_2 + n_3 = 3.$

The 3 particle wave function must be

$$\Phi(x_1\, x_2\, x_3) = \alpha\, u_1(x_1)\, u_1(x_2)\, u_1(x_3)$$

$$+ \beta \frac{1}{\sqrt{6}} \{ u_2(x_1) u_1(x_2) u_0(x_3) + u_1(x_1) u_2(x_2) u_0(x_3) + u_1(x_1) u_0(x_2) u_2(x_3) + u_2(x_1) u_0(x_2) u_1(x_3) + u_0(x_1) u_2(x_2) u_1(x_3) + u_0(x_1) u_1(x_2) u_2(x_3) \}$$

$$+ \gamma \frac{1}{\sqrt{3}} \{ u_3(x_1) u_0(x_2) u_0(x_3) + u_0(x_1) u_0(x_2) u_3(x_3) + u_0(x_1) u_3(x_2) u_0(x_3) \}$$

n_1	n_2	n_3
1	1	1
2	1	0
3	0	0

2 POINTS

where normalization requires $\alpha^2 + \beta^2 + \gamma^2 = 1.$

Problem 12

$$\Phi(\vec{x}_1\ \vec{x}_2) = \frac{1}{\sqrt{2}}\left[u_{p_1}(x_1)\,u_{p_2}(x_2) + u_{p_2}(x_1)\,u_{p_1}(x_2)\right]$$

assuming $\vec{p}_2 \neq \vec{p}_1$

where $u_p(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{x}}$ (finite volume normalization)

(a) First, calculate $\langle \overbrace{\vec{p}_1{}'\ \vec{p}_2{}'}^{b} | V_2 | \overbrace{\vec{p}_1\ \vec{p}_2}^{a} \rangle$

$$= \frac{1}{2V^2}\int d^3x_1\, d^3x_2 \left\{ e^{-ip_1'x_1} e^{-ip_2'x_2} + e^{-ip_2'x_1} e^{-ip_1'x_2} \right\} V(\vec{x}_1, \vec{x}_2) \left\{ e^{ip_1x_1} e^{ip_2x_2} + e^{ip_2x_1} e^{ip_1x_2} \right\}$$

$$= \frac{1}{2V^2}\int d^3x_1\, d^3x_2 \left\{ e^{i(p_1-p_1')x_1} e^{i(p_2-p_2')x_2} + e^{i(p_2-p_2')x_1} e^{i(p_1-p_1')x_2} + e^{i(p_2-p_1')x_1} e^{i(p_1-p_2')x_2} + e^{i(p_1-p_2')x_1} e^{i(p_2-p_1')x_2} \right\} V(\vec{x}_1, \vec{x}_2)$$

Introduce center of mass coordinate $\vec{R} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$
and relative coordinate $\vec{r} = \vec{x}_1 - \vec{x}_2$;
or, $\vec{x}_1 = \vec{R} + \frac{1}{2}\vec{r}$ and $\vec{x}_2 = \vec{R} - \frac{1}{2}\vec{r}$.

$$\langle b | V_2 | a \rangle = \frac{1}{2V^2}\int d^3R\, d^3r\ V(\vec{x}_1, \vec{x}_2) \left\{ e^{i(p_1+p_2-p_1'-p_2')\cdot\vec{R}} \right\} \left\{ e^{i(p_1-p_1'-p_2+p_2')\cdot\vec{r}/2} + e^{i(p_2-p_2'-p_1+p_1')\cdot\vec{r}/2} + e^{i(p_2-p_1'-p_1+p_2')\cdot\vec{r}/2} + e^{i(p_1-p_2'-p_2+p_1')\cdot\vec{r}/2} \right\}$$

Now let $b = a$; i.e., $\vec{p}_1{}' = \vec{p}_1$ and $\vec{p}_2{}' = \vec{p}_2$. Also,
Let $\vec{p}_1 - \vec{p}_2 = \vec{K}$. Then

$$\langle a | V_2 | a \rangle = \frac{1}{2V^2}\int d^3R\, d^3r\ V_2(\vec{x}_1, \vec{x}_2) \left\{ 2 + e^{-i\vec{K}\cdot\vec{r}} + e^{i\vec{K}\cdot\vec{r}} \right\}$$

2 POINTS

(b) Now suppose $V_2(\vec{x}_1, \vec{x}_2) = U(\vec{x}_1 - \vec{x}_2) = U(\vec{r})$.

Then

$$\langle a | U | a \rangle = \frac{1}{2V^2} \int d^3R \, d^3r \, U(\vec{r}) \left[2 + e^{i\vec{K}\cdot\vec{r}} + e^{-i\vec{K}\cdot\vec{r}} \right]$$

$$= \frac{1}{2V} \left\{ 2\hat{U}(\vec{0}) + \hat{U}(\vec{K}) + \hat{U}(-\vec{K}) \right\} \qquad \vec{K} = \vec{p}_1 - \vec{p}_2$$

where $\hat{U}(\vec{K}) = \int d^3r \, e^{i\vec{K}\cdot\vec{r}} \, U(\vec{r})$ = Fourier transform

CHECK If $U(\vec{r})$ = constant (C) then $\langle a | U | a \rangle = C$ for $\vec{K} \neq 0$. ✓

2 POINTS

Problem 13 To prove: $[\Omega, N] = 0$ for fermions.

Recall $N = \int \psi^+(\vec{x})\psi(\vec{x})\, d^3x$

and $\Omega = \int \psi^+(\vec{x}_1)\psi^+(\vec{x}_2) V_2(\vec{x}_1,\vec{x}_2)\psi(\vec{x}_2)\psi(\vec{x}_1)\, d^3x_1\, d^3x_2$,

where $\psi(\vec{x})$ = the fermion field; I suppress the spin index.

So $\{\psi(\vec{x}), \psi(\vec{y})\} = 0$ and $\{\psi(\vec{x}), \psi^+(\vec{y})\} = \delta^3(\vec{x}-\vec{y})$.

Use these identities

$$[AB, C] = A[B,C] + [A,C]B$$

$$[AB, C] = A\{B,C\} - \{A,C\}B$$

$$[N,\Omega] = \underbrace{\int d^3x\, d^3x_1\, d^3x_2\, V_2(\vec{x}_1\vec{x}_2)}_{\text{denote this by } F} \left[\psi_x^+\psi_x, \psi_1^+\psi_2^+\psi_2\psi_1\right]$$

$$= F\left(\psi_x^+\left[\psi_x, \psi_1^+\psi_2^+\psi_2\psi_1\right] + \left[\psi_x^+, \psi_1^+\psi_2^+\psi_2\psi_1\right]\psi_x\right)$$

$$= F\Big(\psi_x^+\left(\left[\psi_x, \psi_1^+\psi_2^+\right]\psi_2\psi_1 + \psi_1^+\psi_2^+\overbrace{\left[\psi_x, \psi_2\psi_1\right]}^{0}\right)$$

$$+ \left(\underbrace{\left[\psi_x^+, \psi_1^+\psi_2^+\right]}_{0}\psi_2\psi_1 + \psi_1^+\psi_2^+\left[\psi_x^+, \psi_2\psi_1\right]\right)\psi_x\Big)$$

Note: $[\psi_x, \psi_2\psi_1] = 0$ and $[\psi_x^+, \psi_1^+\psi_2^+] = 0$.

Also: $[\psi_x, \psi_1^+\psi_2^+] = \{\psi_x, \psi_1^+\}\psi_2^+ - \psi_1^+\{\psi_x, \psi_2^+\}$

$$= \delta^3(\vec{x}-\vec{x}_1)\psi_2^+ - \delta^3(\vec{x}-\vec{x}_2)\psi_1^+$$

and $[\psi_x^+, \psi_2\psi_1] = \delta^3(x-x_2)\psi_1 - \delta^3(\vec{x}-\vec{x}_1)\psi_2$

$$[N,\Omega] = F\Big(\delta^3(x-x_1)\left(\psi_x^+\psi_2^+\psi_2\psi_1 - \psi_1^+\psi_2^+\psi_2\psi_x\right) \leftarrow \vec{x}=\vec{x}_1$$

$$+ \delta^3(x-x_2)\left(-\psi_x^+\psi_1^+\psi_2\psi_1 + \psi_1^+\psi_2^+\psi_1\psi_x\right)\Big) \leftarrow \vec{x}=\vec{x}_2$$

$= 0$ QED

2 POINTS

Problem 14 Consider $\psi(\vec{x},t)$ fermionic; suppress the spin index

In the Heisenberg picture,

$$i\hbar\frac{\partial\psi}{\partial t} = -[H,\psi] \longleftarrow \equiv F[\psi]$$

where $H = H_1 + H_2$

$$\begin{cases} H_1 = \int \psi_\xi^+ t_\xi \psi_\xi \, d^3\xi \\ H_2 = \frac{1}{2}\int \psi_1^+\psi_2^+ V_2(\vec{x}_1\vec{x}_2)\psi_2\psi_1 \, d^3x_1\, d^3x_2 \end{cases}$$

First term

$$[H_1, \psi(x)] = \int d^3\xi \, [\psi_\xi^+ t_\xi \psi_\xi, \psi_x]$$

$$= \int d^3\xi \left(\psi_\xi^+ \underbrace{\{t_\xi\psi_\xi, \psi_x\}}_{0} - \underbrace{\{\psi_\xi^+, \psi_x\}}_{\delta^3(\xi-x)} t_\xi\psi_\xi \right)$$

$$= -t_x\psi_x$$

2nd term

$$[H_2, \psi(\vec{x})] = \frac{1}{2}\int d^3x_1\, d^3x_2\, V_2(\vec{x}_1\vec{x}_2)\,[\psi_1^+\psi_2^+\psi_2\psi_1, \psi_x]$$

$$= \frac{1}{2}\int d^3x_1\, d^3x_2\, V_2 \left(\psi_1^+\psi_2^+ \underbrace{[\psi_2\psi_1, \psi_x]}_{0} + [\psi_1^+\psi_2^+, \psi_x]\psi_2\psi_1 \right)$$

$$= \frac{1}{2}\int d^3x_1\, d^3x_2\, V_2^{(x_1x_2)} \left(\psi_1^+ \underbrace{\{\psi_2^+, \psi_x\}}_{\delta^3(x-x_2)} - \underbrace{\{\psi_1^+, \psi_x\}}_{\delta^3(x-x_1)} \psi_2^+ \right) \psi_2\psi_1$$

$$= \frac{1}{2}\int d^3y \left(V_2(\vec{y},\vec{x})\,\psi_y^+\psi_x\psi_y - V_2(\vec{x},\vec{y})\,\psi_y^+\psi_y\psi_x \right)$$

$$= -\frac{1}{2}\int d^3y\, V(\vec{x},\vec{y})\,\psi_y^+ \left(\underbrace{\psi_y\psi_x - \psi_x\psi_y}_{= 2\psi_y\psi_x} \right)$$

Result

$$F[\psi] = t_x\psi(\vec{x}) + \int d^3y\, V(\vec{x},\vec{y})\,\psi^+(y)\left(\psi(\vec{y})\,\psi(\vec{x})\right)$$

Note : $F[\psi]$ is NONLINEAR.

$$F[\psi] = t_x\psi(\vec{x}) + \int d^3x\, V(\vec{x},\vec{y})\,\psi^+(y)\,\psi(y)\,\psi(\vec{x})$$

$\boxed{\textbf{2 POINTS}}$

Problem 15

For fermions, calculate

$$\langle 0 | c_\alpha c_\beta \psi^+(\vec{r}_1) \psi^+(\vec{r}_2) | 0 \rangle . \quad \equiv Q_{\alpha\beta}(\vec{r}_1, \vec{r}_2)$$

$$\psi(\vec{r}) = \sum_\gamma u_\gamma(\vec{r})\, c_\gamma$$

$$\psi^+(\vec{r}) = \sum_\gamma u_\gamma^*(\vec{r})\, c_\gamma^+$$

So

$$= - c_\gamma^+ c_\beta + \{c_\beta, c_\gamma^+\}$$

$$Q = \sum_\gamma u_\gamma^*(\vec{r}_1) \sum_\delta u_\delta^*(\vec{r}_2) \langle 0 | c_\alpha \underbrace{c_\beta c_\gamma^+} c_\delta^+ | 0 \rangle$$

$$= \sum_{\gamma\delta} u_\gamma^*(r_1)\, u_\delta^*(r_2) \left\{ \langle 0 | c_\alpha \left[-c_\gamma^+ c_\beta + \delta(\beta,\gamma) \right] c_\delta^+ | 0 \rangle \right\}$$

$$= - \sum_{\gamma\delta} u_\gamma^*(r_1)\, u_\delta^*(r_2) \langle 0 | c_\alpha c_\gamma^+ c_\beta c_\delta^+ | 0 \rangle$$

$$+ \sum_\delta u_\beta^*(r_1)\, u_\delta^*(r_2) \langle 0 | c_\alpha c_\delta^+ | 0 \rangle$$

Note: $\langle 0 | c_\gamma^+ = 0$ and $c_\beta | 0 \rangle = 0$

and $c_\alpha | 0 \rangle = 0 \quad \Longrightarrow$

$$Q = - \sum_{\gamma\delta} u_\gamma^*(r_1)\, u_\delta^*(r_2)\, \delta_{\alpha\gamma}\, \delta_{\beta\delta} + \sum_\delta u_\beta^*(r_1)\, u_\delta^*(r_2)\, \delta_{\alpha\delta}$$

$$Q = u_\beta^*(\vec{r}_1)\, u_\alpha^*(\vec{r}_2) - u_\alpha^*(\vec{r}_1)\, u_\beta^*(\vec{r}_2)$$

2 POINTS

Note $Q_{\alpha\beta}(\vec{r}_1 \vec{r}_2)$ is antisymmetric w.r.t. exchange of (α,β), ~~and~~ exchange of $(\vec{r}_1, \vec{r}_2)$.