Homework Assignment #4

Problem 16 *Equal time commutation relations.*

2 POINTS

For the second quantized Schroedinger equation, we have, in the Schroedinger picture,

$$[\,\psi(\mathbf{x}) , \psi^\dagger(\mathbf{x}')\,] = \delta^3(\mathbf{x} - \mathbf{x}') \quad ,$$

$$[\,\psi(\mathbf{x}) , \psi(\mathbf{x}')\,] = 0 \text{ , etc.}$$

(a) Show that in the Heisenberg picture, this commutation relation holds at all equal times.

Proof

$$\psi_h(\mathbf{x},t) = e^{iHt}\ \psi(\mathbf{x})\ e^{-iHt}$$

$$\psi_h^\dagger(\mathbf{x},t) = e^{iHt}\ \psi^\dagger(\mathbf{x})\ e^{-iHt}$$

$$[\,\psi_h(\mathbf{x},t) , \psi_h^\dagger(\mathbf{x}',t)\,] = [\,e^{iHt}\ \psi(\mathbf{x})\ e^{-iHt} ,\ e^{iHt}\ \psi^\dagger(\mathbf{x}')\ e^{-iHt}\,]$$

$$= e^{iHt}\ [\,\psi(\mathbf{x})\ ,\ \psi^\dagger(\mathbf{x}')\,]\ e^{-iHt}$$

$$= \delta^3(\mathbf{x} - \mathbf{x}') \qquad \text{which does not depend on t.}$$

(b) What is the commutation relation for *different* times?

$$[\,\psi_h(\mathbf{x},t) , \psi_h^\dagger(\mathbf{x}',t')\,] = [\,e^{iHt}\ \psi(\mathbf{x})\ e^{-iHt} ,\ e^{iHt'}\ \psi^\dagger(\mathbf{x}')\ e^{-iHt'}\,]$$

$$= e^{iHt}\ \psi(\mathbf{x})\ e^{-iH(t-t')}\ \psi^\dagger(\mathbf{x}')\ e^{-iHt'}$$

$$-\ e^{iHt'}\ \psi^\dagger(\mathbf{x}')\ e^{+iH(t-t')}\ \psi(\mathbf{x})\ e^{-iHt}$$

which is hard to simplify.

Problem 17(a) = **MANDL and SHAW, problem 2.1**

Consider a field theory with a set of fields

$\{ \varphi_1(x), \varphi_2(x), \ldots, \varphi_r(x), \ldots, \varphi_N(x) \}$.

First consider Lagrangian density

$\pounds = \pounds(\varphi_r(x), \partial_\alpha\varphi_r(x)) = \pounds(\varphi_r, \varphi_{r,\alpha})$

The field equations (i.e., Euler-Lagrangian equations) are

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}}\right) = 0 \qquad (r = 1\,2\,3 \ldots N)$$

Now modify the Lagrangian density. Let

$\pounds' = \pounds + \partial_\alpha \Lambda^\alpha$

where $\Lambda^\alpha(x)$ is a function of $\{ \varphi_1(x), \varphi_2(x), \ldots, \varphi_r(x), \ldots, \varphi_N(x) \}$.

Now what are the field equations?

$$\frac{\partial \mathcal{L}'}{\partial \phi_r} - \frac{\partial}{\partial x^\beta}\left(\frac{\partial \mathcal{L}'}{\partial \phi_{r,\alpha}}\right) = 0$$

Note:

$$\partial_\alpha \Lambda^\alpha = \sum_{s=1}^{N} \frac{\partial \Lambda^\alpha}{\partial \phi_s}\frac{\partial \phi_s}{\partial x^\alpha} = \sum_{s=1}^{N} \frac{\partial \Lambda^\alpha}{\partial \phi_s}\phi_{s,\alpha}$$

2 POINTS

$$\hookrightarrow \; = \frac{\partial \mathcal{L}}{\partial \phi_r} + \frac{\partial}{\partial \phi_r}\left(\frac{\partial \Lambda^\alpha}{\partial x^\alpha}\right) - \frac{\partial}{\partial x^\beta}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\beta}}\right) - \underbrace{\frac{\partial}{\partial x^\beta}\left[\frac{\partial}{\partial \phi_{r,\beta}}\left(\frac{\partial \Lambda^\alpha}{\partial x^\alpha}\right)\right]}$$

$$= \frac{\partial}{\partial x^\beta}\frac{\partial}{\partial \phi_{r,\beta}}\sum_{s=1}^{N}\frac{\partial \Lambda^\alpha}{\partial \phi_s}\phi_{s,\alpha}$$

$$= \frac{\partial}{\partial x^\beta}\sum_{s=1}^{N}\frac{\partial \Lambda^\alpha}{\partial \phi_s}\delta_{rs}\delta_{\beta\alpha}$$

$$= \frac{\partial}{\partial x^\alpha}\frac{\partial \Lambda^\alpha}{\partial \phi_r} = \sum_{s=1}^{N}\frac{\partial^2 \Lambda^\alpha}{\partial \phi_r \partial \phi_s}\frac{\partial \phi_s}{\partial x^\alpha}$$

$$= \frac{\partial}{\partial \phi_r}\left(\frac{\partial \Lambda^\alpha}{\partial x^\alpha}\right)$$

$$\frac{\partial \mathcal{L}'}{\partial \phi_r} - \frac{\partial}{\partial x^\beta}\left(\frac{\partial \mathcal{L}'}{\partial \phi_{r,\alpha}}\right) = \frac{\partial \mathcal{L}}{\partial \phi_r} - \frac{\partial}{\partial x^\beta}\left(\frac{\partial \mathcal{L}}{\partial \phi_{r,\beta}}\right) = 0$$

$\mathcal{L}'$ and $\mathcal{L}$ imply the same equat'ns of motion.

Problem 17(b) = **MANDL and SHAW, problem 2.2**

The real scalar field is $\varphi(x)$ and the Hamiltonian density is

$$\mathcal{H}(x) = \tfrac{1}{2}\left\{ c^2\,\Pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2 \right\}$$

The *equal-time commutation relations* for the quantized fields are

$$[\,\varphi(x)\,,\,\Pi(x')\,] = i\hbar\,\delta^3(\,\mathbf{x}-\mathbf{x'}\,)$$

$$[\,\varphi(x)\,,\,\varphi(x')\,] = 0 \qquad \text{and} \qquad [\,\Pi(x)\,,\,\Pi(x')\,] = 0$$

- Calculate $[\,H\,,\ \varphi(x)\,]$

$$= \frac{c^2}{2}\int d^3x'\,\left[\,\pi^2(x'),\,\phi(x)\right] \qquad (t'=t)$$

$$= \frac{c^2}{2}\int d^3x'\left\{\,\pi(x')\left[\pi(x'),\phi(x)\right] + \left[\pi(x'),\phi(x)\right]\pi(x')\,\right\}$$

$$= -i\hbar c^2\,\pi(x)$$

- Calculate $[\,H\,,\ \Pi(x)\,]$

$$= \frac{1}{2}\int d^3x'\left\{\,\underbrace{\left[\,(\nabla\phi(x'))^2,\ \pi(x)\right]}_{i\hbar\,(-\nabla^2)\,\phi(x)\times 2} + \mu^2\underbrace{\left[\phi^2(x'),\ \pi(x)\right]}_{2i\hbar\,\phi(x)\times 2}\right\}$$

$$= 2i\hbar\left(-\nabla^2+\mu^2\right)\phi(x)$$

2 POINTS

Next, the Heisenberg equations of motion are

$$\frac{\partial\phi}{\partial t} = \frac{i}{\hbar}\left[H,\phi\right] = c^2\pi \qquad \longleftarrow \dot{\phi} = c^2\pi$$

$$\frac{\partial\pi}{\partial t} = \frac{i}{\hbar}\left[H,\pi\right] = \left(\nabla^2-\mu^2\right)\phi$$

Combining the equations,

Combining the equations,

$$\ddot{\phi} = c^2\,\dot{\pi} = c^2\left(\nabla^2-\mu^2\right)\phi$$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + \mu^2\phi = 0$$

$$(\Box+\mu^2)\,\phi = 0. \qquad \longleftarrow (\Box+\mu^2)\phi = 0$$

2 POINTS

<u>Problem 17(c)</u> = **MANDL and SHAW, problem 2.3**

Consider a real vector field $\varphi^\alpha(x)$ with this Lagrangian density ,

$$\pounds = -\tfrac{1}{2} (\partial_\alpha \varphi_\beta) (\partial^\alpha \varphi^\beta) + \tfrac{1}{2} (\partial_\alpha \varphi^\alpha) (\partial_\beta \varphi^\beta) + \tfrac{1}{2} \mu^2 \varphi_\alpha \varphi^\alpha$$

or

$$\pounds = -\tfrac{1}{2} (\varphi_{\beta,\alpha}) (\varphi^{\beta,\alpha}) + \tfrac{1}{2} (\varphi^\alpha{}_{,\alpha}) (\varphi^\beta{}_{,\beta}) + \tfrac{1}{2} \mu^2 \varphi_\alpha \varphi^\alpha$$

where $\varphi_{\beta,\alpha} = \partial_\alpha \varphi_\beta$.

- Derive the field equation (Euler-Lagrange equation)

$$\frac{\partial \mathcal{L}}{\partial \phi^\mu} - \frac{\partial}{\partial x^\lambda}\left(\frac{\partial \mathcal{L}}{\partial \phi^\mu{}_{,\lambda}}\right) = 0$$

- $\frac{\partial \mathcal{L}}{\partial \phi^\mu} = \frac{\partial}{\partial \phi^\mu} \frac{1}{2}\mu^2 g_{\alpha\beta} \phi^\alpha \phi^\beta$

$$= \frac{1}{2}\mu^2 g_{\alpha\beta} \{ \delta^\alpha_\mu \phi^\beta + \delta^\beta_\mu \phi^\alpha \} = \mu^2 \phi_\mu$$

- $\frac{\partial \mathcal{L}}{\partial \phi^\mu{}_{,\lambda}} = -\phi_\mu{}^{,\lambda} + \delta^\alpha_\mu \delta^\lambda_\alpha \phi^\beta{}_{,\beta}$

$$= -\phi_\mu{}^{,\lambda} + \delta^\lambda_\mu (\phi^\beta{}_{,\beta})$$

So the field equation is

$$\mu^2 \phi_\mu - \frac{\partial}{\partial x^\lambda}\left[-\frac{\partial \phi_\mu}{\partial x_\lambda} + \delta^\lambda_\mu \frac{\partial \phi^\beta}{\partial x^\beta} \right] = 0$$

$$\mu^2 \phi_\mu + \Box \phi_\mu - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \phi^\beta}{\partial x^\beta}\right) = 0$$

$$g_{\alpha\beta} (\Box + \mu^2) \phi^\beta - \partial_\alpha \partial_\beta \phi^\beta = 0$$

- Also, prove the Lorentz condition: $\partial_\alpha \varphi^\alpha = 0$.

2 POINTS

$$(\Box + \mu^2) \phi_\alpha - \partial_\alpha (\partial_\beta \phi^\beta) = 0$$

$$\partial^\alpha [(\Box + \mu^2) \phi_\alpha] - \partial^\alpha \partial_\alpha (\partial_\beta \phi^\beta) = 0$$

$$\Box (\partial^\alpha \phi_\alpha) + \mu^2 \partial^\alpha \phi_\alpha - \Box (\partial_\alpha \phi^\alpha) = 0$$

Thus $\partial_\alpha \phi^\alpha = 0$

2 POINTS

Problem 18

For the free real scalar field (Section 3.1)

(A) Write H in terms of $\Pi(x)$ and $\varphi(x)$.

(A) $L = \int d^3x \, \frac{1}{2}\left[\dot\phi^2 - (\nabla\phi)^2 - \mu^2\phi^2 \right]$ setting $\hbar = 1$ and $c = 1$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$$

$H = \int d^3x \left[\pi\dot\phi - \mathcal{L} \right]$ rewritten in terms of π and ϕ

$$H = \int d^3x \left[\pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 \right]$$

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 \right]$$

1 POINT

(B) Write H in terms of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$.

(B) $\phi(\vec x, t) = \sum_{\vec k} \left(\frac{1}{2\omega\Omega}\right)^{1/2} \left\{ a_{\vec k}\, e^{i(\vec k\cdot\vec x - \omega t)} + a_{\vec k}^\dagger\, e^{-i(\vec k\cdot\vec x - \omega t)} \right\}$ where $\omega = \sqrt{k^2 + \mu^2}$

$$\dot\phi = \pi = \sum_{\vec k} \left(\frac{1}{2\omega\Omega}\right)^{1/2} (-i\omega) \left\{ a_{\vec k}\, e^{i(\vec k\cdot\vec x - \omega t)} - a_{\vec k}^\dagger\, e^{-i(\vec k\cdot\vec x - \omega t)} \right\}$$

Now set $t = 0$ and calculate H. (H does not depend on t.)

$$H = \int d^3x \sum_{\vec k_1}\sum_{\vec k_2} \left(\frac{1}{2\omega_1 2\omega_2}\right)^{1/2} \frac{1}{2\Omega} \Big\{ -\omega_1\omega_2 \left[a_{\vec k_1} e^{i\vec k_1\cdot\vec x} - a_{\vec k_1}^\dagger e^{-i\vec k_1\cdot\vec x} \right]\left[a_{k_2} e^{i\vec k_2\cdot\vec x} - a_{k_2}^\dagger e^{-i\vec k_2\cdot\vec x} \right]$$
$$+ (-\vec k_1\cdot\vec k_2)\left[a_{\vec k_1} e^{i\vec k_1\cdot\vec x} - a_{\vec k_1}^\dagger e^{-i\vec k_1\cdot\vec x} \right]\left[a_{\vec k_2} e^{i\vec k_2\cdot\vec x} - a_{\vec k_2}^\dagger e^{-i\vec k_2\cdot\vec x} \right]$$
$$+ \mu^2 \left[a_{k_1} e^{i\vec k_1\cdot\vec x} + a_{\vec k_1}^\dagger e^{-i\vec k_1\cdot\vec x} \right]\left[a_{k_2} e^{i\vec k_2\cdot\vec x} + a_{k_2}^\dagger e^{-i\vec k_2\cdot\vec x} \right] \Big\}$$

$\int d^3x\, e^{i\vec k_1\cdot\vec x} e^{i\vec k_2\cdot\vec x} = \delta_K(\vec k_1, -\vec k_2)$; $\int d^3x\, e^{i\vec k_1\cdot\vec x} e^{-i\vec k_2\cdot\vec x} = \delta_K(\vec k_1, \vec k_2)$; etc

$$H = \frac{1}{\Omega}\sum_{\vec k_1} \frac{1}{2\omega_1}\, \frac{1}{2} \left\{ a_{\vec k_1} a_{-\vec k_1} \left(-\omega_1^2 + \vec k_1^2 + \mu^2\right) + a_{k_1}^\dagger a_{-\vec k_1}^\dagger \left(-\omega_1^2 + \vec k_1^2 + \mu^2\right) + \left(a_{\vec k_1} a_{\vec k_1}^\dagger + a_{k_1}^\dagger a_{k_1} \right)\left(\omega_1^2 + \vec k_1^2 + \mu^2 \right) \right\}$$

1 POINT

$$H = \int \frac{d^3k_1}{(2\pi)^3}\, \frac{1}{2} \left(a_{\vec k_1} a_{\vec k_1}^\dagger + a_{\vec k_1}^\dagger a_{\vec k_1} \right) \hbar\omega_{\vec k_1} = \int \frac{d^3k}{(2\pi)^3}\, \hbar\omega_{\vec k} \left(a_{\vec k}^\dagger a_{\vec k} + \frac{1}{2} \right)$$

Problem 19(A) = Mandl and Shaw problem 3.1.

Problem 19(A) Mandl & Shaw problem 3.1

- Start with equation 3.7:

$\omega' t - \vec{k}'\cdot\vec{x}$

$$\phi(\vec{x},t) = \sum_{\vec{k}'}\left(\frac{1}{2\omega\Omega}\right)^{1/2}\left[a_{\vec{k}'}e^{-ik'\cdot x} + a^{+}_{\vec{k}'}e^{ik'\cdot x}\right]$$

Now consider

$$\int d^3x\; e^{ik\cdot x}\left[i\dot{\phi}(x) + \omega_{\vec{k}}\,\phi(x)\right]$$

$$= \sum_{\vec{k}'}\left(\frac{1}{2\omega\Omega}\right)^{1/2}\Big[i(-i\omega')\,a_{\vec{k}'}\,e^{i(\omega-\omega')t}\,\Omega\delta_K(\vec{k}',k) \quad \leftarrow \text{combines}$$

$$+\, i(i\omega')\,a^{+}_{\vec{k}'}\,e^{i(\omega+\omega')t}\,\Omega\delta_K(\vec{k}',-\vec{k}) \quad \leftarrow \text{cancels}$$

$$+\, \omega_k\, a_{\vec{k}'}\,e^{i(\omega-\omega')t}\,\Omega\delta_K(\vec{k}',\vec{k}) \quad \leftarrow \text{combines}$$

$$+\, \omega_{\vec{k}}\, a^{+}_{\vec{k}'}\,e^{i(\omega+\omega')t}\,\Omega\delta_K(\vec{k}',-\vec{k})\Big] \quad \leftarrow \text{cancels}$$

$$= \left(\frac{1}{2\omega\Omega}\right)^{1/2} 2\omega\Omega\, a_{\vec{k}} = (2\omega\Omega)^{1/2}\, a_{\vec{k}}$$

So

$$a_{\vec{k}} = \frac{1}{(2\omega\Omega)^{1/2}}\int d^3x\; e^{ik\cdot x}\left[i\dot{\phi} + \omega\phi\right]$$

1 POINT

CAN SET $t=0$ because $a_{\vec{k}}$ does not depend on t

- Now, for the quantized field, calculate

$$[a_{\vec{k}},\, a^{+}_{\vec{k}'}] = \left(\frac{1}{2\omega\Omega\; 2\omega'\Omega}\right)^{1/2}\int d^3x_1\, d^3x_2\; e^{ik\cdot x_1}\, e^{-ik'\cdot x_2}$$

$$\left[\left[i\dot{\phi} + \omega\phi\right]_{x_1},\ \left(-i\dot{\phi} + \omega'\phi\right)_{x_2}\right] \quad \text{EQUAL TIMES}$$

$$= \left(\frac{1}{2\omega\, 2\omega'}\right)^{1/2}\frac{1}{\Omega}\int d^3x_1\, d^3x_2\; e^{-i\vec{k}\cdot\vec{x}_1}\, e^{+i\vec{k}'\cdot\vec{x}_2}\left\{-i\omega_k\, i\,\delta^3(\vec{x}_1-\vec{x}_2) - i\omega_{k'}(-i)\,\delta^3(\vec{x}_1-\vec{x}_2)\right\}$$

$$= \frac{\omega+\omega'}{(2\omega\cdot 2\omega')^{1/2}}\,\underbrace{\frac{1}{\Omega}\int d^3x_1\; e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}_1}}_{\frac{1}{\Omega}\Omega\,\delta(\vec{k},\vec{k}')} = \delta(\vec{k},\vec{k}')$$

1 POINT

Problem 19(B) = **Mandl and Shaw problem 3.2.**

Problem 19 (B) Mandl & Shaw problem 3.2

$$\phi_r(x) = \sum_{\vec{k}} \left(\frac{1}{2\omega_k \Omega}\right)^{1/2} \left[a_r(\vec{k}) e^{-ik\cdot x} + a_r^{+}(\vec{k}) e^{ik\cdot x} \right] \quad (r=1,2)$$

(Setting $\hbar = 1$ and $c = 1$)

(★) FIRST PART

Use
$$\begin{cases} \phi(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \sum_{\vec{k}} \left(\frac{1}{2\omega_k \Omega}\right)^{1/2} \left[a_{\vec{k}} e^{-ik\cdot x} + b_k^{+} e^{ik\cdot x} \right] \\ \phi^{+}(x) = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) = \sum_{\vec{k}} \left(\frac{1}{2\omega_k \Omega}\right)^{1/2} \left[b_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^{+} e^{ik\cdot x} \right] \end{cases}$$

implies
$$a_{\vec{k}} = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) + i a_2(\vec{k})\right] \quad \text{and} \quad a_k^{+} = \frac{1}{\sqrt{2}}(a_1^{+} - i a_2^{+})$$
$$b_{\vec{k}} = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - i a_2(\vec{k})\right] \quad \text{and} \quad b_k^{+} = \frac{1}{\sqrt{2}}(a_1^{+} + i a_2^{+})$$

1 POINT

(★) OTHER PART

I'll derive equations (3.27a) and (3.27b):

(3.27a)
$$[a_{\vec{k}}, a_{\vec{k}'}^{+}] = \frac{1}{2}\left[a_1 + i a_2, a_1^{+} - i a_2^{+}\right] = \frac{1}{2}[a_1, a_1^{+}] + \frac{1}{2}[a_2, a_2^{+}]$$
$$= \delta(\vec{k}, \vec{k}')$$

and similarly $[b_{\vec{k}}, b_{\vec{k}'}^{+}] = \delta(\vec{k}, \vec{k}')$ $\Big\}$ (3.27a)

(3.27b)
$$[a_{\vec{k}}, a_{\vec{k}'}] = \frac{1}{2}[\overset{k}{a_1 + i a_2}, \overset{k'}{a_1 + i a_2}] = \frac{1}{2} i [\overset{k}{a_1}, \overset{k'}{a_2}] + \frac{1}{2} i [\overset{k}{a_2}, \overset{k'}{a_1}] = 0 \quad ✓$$

and similarly $[b_{\vec{k}}, b_{\vec{k}'}] = 0$ ✓

$$[a_{\vec{k}}, b_{\vec{k}'}] = \frac{1}{2}[\overset{k}{a_1 + i a_2}, \overset{k'}{a_1 - i a_2}] = 0 \quad \text{because } [a_i, a_j] = 0 \quad ✓$$

$$[a_{\vec{k}}^{+}, b_{\vec{k}'}] = \frac{1}{2}[\overset{k}{a_1^{+} - i a_2^{+}}, \overset{k'}{a_1 - i a_2}] = \frac{1}{2}[-\delta(\vec{k}, \vec{k}')] - \frac{1}{2}[-\delta(\vec{k}, \vec{k}')]$$
$$= 0 \quad ✓$$

$\Big\} \leftarrow$ (3.27b)

1 POINT

Problem 20 = **The Yukawa Theory Problem**

Problem 20 — The Yukawa theory problem

The potential is

correction

$$V(\vec{x}) = -g^2 \frac{3}{4\pi r_0^3 A} \int \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} \theta(r_0 A^{1/3} - |\vec{y}|)\, d^3y \quad (\times A)$$

where I have set $\hbar = 1$ and $c = 1$.

To do the numerical calculation I need to restore the units. Choose these parameter values:

$r_0 = 1.25$ fm ; $mc^2 = 140$ MeV ; $A = 238$; $g = 15$.

Recall
$$\frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2+m^2}$$

So

$$V(\vec{x}) = \frac{-3g^2}{4\pi r_0^3 A} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2+m^2} \underbrace{\int e^{-i\vec{k}\cdot\vec{y}}\,\theta(R-y)\, d^3y}_{\text{Call that } f(\vec{k}).} \quad (\times A)$$

Call that $f(\vec{k})$. By spherical symmetry it only depends on $|\vec{k}|$. W.L.O.G. let $\vec{k} = k\hat{e}_z$.

$$f(\vec{k}) = \int_0^R y^2 dy \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\; e^{-iky\cos\theta}$$

$$= 2\pi \int_0^R y^2 dy\, \frac{2}{ky}\sin(ky) = \frac{4\pi}{k}\int_0^R y\, dy\, \sin(ky)$$

$$= \frac{4\pi}{k}\left(\frac{-\partial}{\partial k}\right)\int_0^R \cos(ky)\, dy = \frac{4\pi}{k}\left(\frac{-\partial}{\partial k}\right)\frac{\sin kR}{k}$$

$$= \frac{4\pi}{k}\left\{\frac{\sin kR}{k^2} - \frac{R\cos kR}{k}\right\} = \frac{4\pi R^2}{k} j_1(kR)$$

spherical Bessel function

Problem 20 = The Yukawa Theory Problem

$$V(\vec{x}) = \frac{-3g^2}{4\pi r_0^3 A} 4\pi R^2 \underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} \frac{j_1(kr)}{k} e^{-i\vec{k}\cdot\vec{x}}}$$ (× A)

Call that $g(r)$. By spherical symmetry it only depends on $r = |\vec{x}|$; W.L.O.G. let $\vec{x} = r\hat{e}_z$.

$$g(r) = \int_0^\infty \frac{2\pi k^2 dk}{(2\pi)^3(k^2+\mu^2)} \frac{j_1(kR)}{k} \underbrace{\int_0^\pi \sin\theta \, d\theta \, e^{-ikr\cos\theta}}_{\frac{2}{kr}\sin(kr) = 2j_0(kr)}$$

spherical Bessel func.

$$= \frac{(2\pi)\cdot 2}{(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2+\mu^2} \frac{j_1(kR)}{k} j_0(kr)$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{k\,dk}{k^2+\mu^2} j_1(kR) j_0(kr)$$

So the potential energy function is

$$V(\vec{x}) = \frac{-3g^2}{4\pi r_0^3 A} 4\pi R^2 \frac{1}{2\pi^2} \int_0^\infty \frac{k\,dk}{k^2+m^2} j_1(kR) j_0(kr)$$ $R = r_0 A^{1/3}$

let $kR = x$.

$$= \frac{-3g^2}{2\pi^2} \frac{1}{r_0 A^{1/3}} \int_0^\infty \frac{\frac{x}{R}\frac{dx}{R}}{(\frac{x}{R})^2+m^2} j_1(x) j_0\left(x\frac{r}{R}\right)$$ $\therefore dk = \frac{1}{R}dx$

$$= \frac{-3g^2}{2\pi^2 R} \int_0^\infty \frac{x\,dx}{x^2+(mR)^2} j_1(x) j_0(x\,r/R)$$ (× A)

<u>RESTORE THE FACTORS of $\hbar$ and c.</u> V must have units of energy.

Problem 20 = **The Yukawa Theory Problem**

$$V(r) = \frac{-3g^2}{2\pi^2}\,\frac{\hbar c}{R}\int_0^\infty \frac{x\,dx}{x^2+K^2}\, j_1(x)\, j_0(xr/R) \quad \mathbf{(\times A)}$$

Parameters with units

$$R = r_0 A^{1/3} = 1.25\text{ fm} \times (238)^{1/3} = 7.75\text{ fm}$$

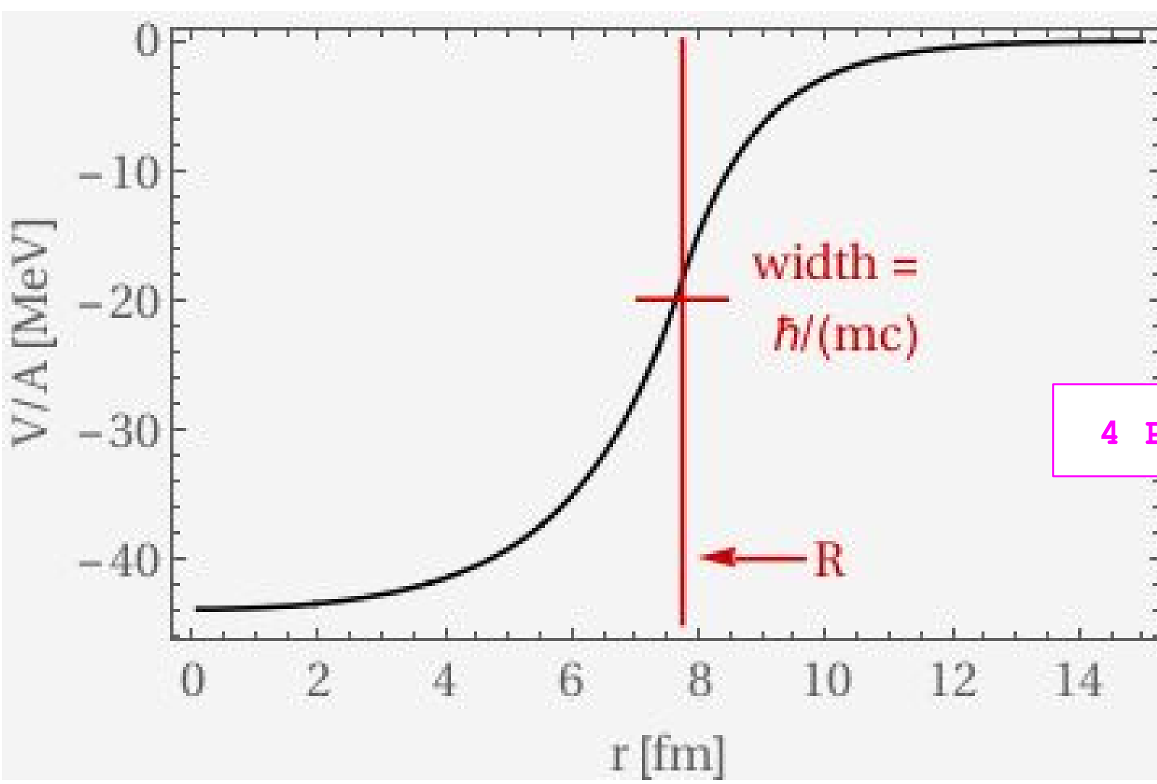
$$\frac{\hbar c}{R} = \frac{197\text{ MeV fm}}{7.75\text{ fm}} = 25.4\text{ MeV}$$

$$g = 15 \;\Rightarrow\; \text{Const.} = 868\text{ MeV}$$

$$\frac{V(r)}{A} = -\,868\text{ MeV}\cdot\int_0^\infty \frac{x\,dx}{x^2+K^2}\, j_1(x)\, j_0(xr/R)$$

where $K = \frac{mc^2 R}{\hbar c} = 5.51$ dimensionless

Do the numerical calculation with MATHEMATICA.



4 POINTS