Homework Problems due Friday Feb 17

Problems 21 and 22 = Mandl and Shaw problems 3.3 and 3.4

(21) Problem 3.3

Using Eq. (3.58), $\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^2-\mu^2+i\epsilon}$

Calculate

$$(\Box+\mu^2)\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{(\Box+\mu^2)e^{-ik\cdot x}}{k^2-\mu^2+i\epsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{(-k^2+\mu^2)e^{-ik\cdot x}}{k^2-\mu^2+i\epsilon}$$

$$= \int \frac{d^4k}{(2\pi)^4} (-)\, e^{-ik\cdot x} \quad \text{because } \epsilon\to 0$$

$$= -\delta^4(x)$$

2 POINTS

(22) Problem 3.4 Consider $\langle 0 | T\, \phi(x)\, \phi^\dagger(y) | 0\rangle$

$$= \theta(x^0-y^0)\,\langle 0 | \sum_{\vec{k}} N\left(e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} b^\dagger_{\vec{k}}\right) \sum_{\vec{k}'} N'\left(e^{ik'\cdot y} a^\dagger_{\vec{k}'} + e^{-ik'\cdot y} b_{\vec{k}'}\right)|0\rangle$$

+ the other time ordering

$$= \theta(x^0-y^0) \sum_{\vec{k}}\sum_{\vec{k}'} NN'\, e^{-ik\cdot x} e^{ik'\cdot y}\, \delta(\vec{k},\vec{k}') \quad (+\text{ o.T.o.})$$

$$= \theta(x^0-y^0) \sum_{\vec{k}} \frac{1}{2\omega\Omega} e^{-ik\cdot(x-y)} \quad \boxed{k^\mu = (\omega,\vec{k}) = (\sqrt{\vec{k}^2+\mu^2},\vec{k})}$$

(+o.T.o.)

$$= \theta(x^0-y^0)\, i\Delta^+(x-y) \quad \text{by Eq. (3.39)}$$

$$+\, \theta(y^0-x^0)\left[-i\Delta^-(x-y)\right]$$

this is the other time ordering

$$= i\Delta_F(x-y) \quad \text{by equation (3.56a).}$$

$$= i\hbar c\, \Delta_F(x-y) \quad \text{restoring } \hbar \text{ and } c.$$

2 POINTS

Problem 23. Mandl and Shaw problem 3.5.

(23) Problem 3.5 Charge conjugation for the complex scalar field is the transformation $\phi(x) \to \phi'(x) \equiv C\phi(x)C^{-1} = \eta_c \phi^\dagger(x)$ where $C^{-1} = C^\dagger$, $C|0\rangle = |0\rangle$, and $\eta_c = e^{i\lambda}$ (phase factor).

$\therefore\ \phi' = \eta_c \phi^\dagger$ and $\phi'^\dagger = \eta_c^* \phi$.

- The Lagrangian density (3.22) is invariant.

1 POINT

PROOF $\mathcal{L} = N\left(\frac{\partial \phi^\dagger}{\partial x^\alpha}\frac{\partial \phi}{\partial x_\alpha} - \mu^2 \phi^\dagger \phi\right)$

N : the "normal product"; see page 42.

$$\mathcal{L}' = N\left(\eta_c^* \frac{\partial \phi}{\partial x^\alpha}\, \eta_c \frac{\partial \phi^\dagger}{\partial x_\alpha} - \mu^2 \eta_c^* \phi\, \eta_c \phi^\dagger\right)$$

$$= N\left(\frac{\partial \phi}{\partial x^\alpha}\frac{\partial \phi^\dagger}{\partial x_\alpha} - \mu^2 \phi\phi^\dagger\right) \quad \text{because } |\eta|^2 = 1$$

$$= \frac{\partial \phi^\dagger}{\partial x_\alpha}\frac{\partial \phi}{\partial x^\alpha} - \mu^2 \phi^\dagger \phi \quad \text{because } N(\phi\phi^\dagger) = N(\phi^\dagger\phi);$$

see page 42.

$= \mathcal{L}$, as claimed.

- The charge current density changes sign.

1 POINT

PROOF $j^\alpha = -iq N\left[\frac{\partial \phi^\dagger}{\partial x_\alpha}\phi - \frac{\partial \phi}{\partial x_\alpha}\phi^\dagger\right]$

$$j'^\alpha = -iq N\left[\eta^* \frac{\partial \phi}{\partial x_\alpha}\eta\phi^\dagger - \eta\frac{\partial \phi^\dagger}{\partial x_\alpha}\eta^*\phi\right]$$

$$= -iqN\left[\frac{\partial \phi}{\partial x_\alpha}\phi^\dagger - \frac{\partial \phi^\dagger}{\partial x_\alpha}\phi\right] \quad \text{because } |\eta|^2 = 1$$

$$= +iqN\left[\frac{\partial \phi^\dagger}{\partial x_\alpha}\phi - \frac{\partial \phi}{\partial x_\alpha}\phi^\dagger\right]$$

$= -j^\alpha$, as claimed.

Problem 23 continued ...

• Show that $C a_{\vec{k}} C^{-1} = \eta_c b_{\vec{k}}$ and $C b_{\vec{k}} C^{-1} = \eta_c^* a_{\vec{k}}$

<u>Proof</u> We have

$$\phi(x) = \sum_{\vec{k}} N_{\vec{k}} \left[a_{\vec{k}} e^{-ik\cdot x} + b_{\vec{k}}^{+} e^{ik\cdot x} \right] \quad \text{where} \quad N_{\vec{k}} = \sqrt{\frac{1}{2\omega\Omega}}$$

$$\phi'(x) = C\,\phi(x)\,C^{-1} = \eta\,\phi^{+}$$

$$= \sum_{\vec{k}} N_{\vec{k}} \left[C a_{k} C^{-1} e^{-ik\cdot x} + C b_{k}^{+} C\, e^{+ik\cdot x} \right] = \eta \sum_{\vec{k}} N_{k} \left[a_{\vec{k}}^{+} e^{ik\cdot x} + b_{\vec{k}} e^{-ik\cdot x} \right]$$

1 POINT

equate coefficients of $e^{-ik\cdot x} \Rightarrow C a_{\vec{k}} C^{-1} = \eta\, b_{\vec{k}}$ (as claimed)

equate coefficients of $e^{+ik\cdot x} \Rightarrow C b_{\vec{k}}^{+} C^{-1} = \eta\, a_{\vec{k}}^{+}$

or $(C b_{k}^{+} C^{+})^{+} = \eta^{*} a_{\vec{k}} = C b_{\vec{k}} C^{+}$ (as claimed).

• $C\,|a\,;\vec{k}\rangle = \eta_c^{*}\,|b\,;\vec{k}\rangle$

<u>Proof</u> $C\,|a\,;\vec{k}\rangle = C\, a_{\vec{k}}^{+}\,|0\rangle = C a_{\vec{k}}^{+} C^{-1} C\,|0\rangle$

because $C^{-1}C = 1$

$= C a_{\vec{k}}^{+} C^{+}\,|0\rangle$ because $C|0\rangle = |0\rangle$

$= (C a_{\vec{k}} C^{+})^{+}\,|0\rangle = (\eta_c b_{\vec{k}})^{+}\,|0\rangle$

1 POINT

$= \eta_c^{*}\, b_{\vec{k}}^{+}\,|0\rangle = \eta_c^{*}\,|b\,;\vec{k}\rangle$ as claimed.

Problem 24.

A. Determine the Dirac spinors $v_1(p)$ and $v_2(p)$ for antiparticles.

B. Determine the polarization sum $\Lambda^-(p)$ for antiparticles.

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(24) A: Referring to Appendix A, we require

$(\not{p} + mc)\, v_r(p) = 0$ for $r = 1, 2$.

(Notation: $\not{p} = \gamma_\mu p^\mu$)

Now $(\not{p} + mc)(\not{p} - mc) = p^2 - m^2 = 0$.

So $v_r(\vec{p})$ can be a column of $\not{p} - mc$.

$$\not{p} - mc = \gamma^0 \frac{E}{c} - \vec{\gamma}\cdot\vec{p} - mc = \left(\begin{array}{c|c} E/c - mc & -\vec{\sigma}\cdot\vec{p} \\ \hline \vec{\sigma}\cdot\vec{p} & -E/c - mc \end{array}\right)$$

2×2 blocks

We choose the 3rd and 4th columns; drop the minus signs.

$$\left[v_2(\vec{p}), \; v_1(\vec{p}) \right] = \left[\begin{array}{c|c} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \\ E/c + mc & 0 \\ 0 & E/c + mc \end{array}\right]$$

Also, normalize the spinors by $v^\dagger v = \frac{E}{mc^2}$.

RESULTS

$$v_1(\vec{p}) = \sqrt{\frac{E/mc^2}{\vec{p}^2 + (E/c + mc)^2}} \begin{bmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{bmatrix} = \frac{1}{\sqrt{2m(E + mc^2)}} \begin{bmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{bmatrix}$$

$$\frac{E/m}{\vec{p}^2 + (E+m)^2} = \frac{E/m}{E^2 - m^2 + (E+m)^2} = \frac{E/m}{2E^2 + 2Em} = \frac{1}{2m(E+m)}$$

and similarly

$$v_2(\vec{p}) = \frac{1}{\sqrt{2m(E + mc^2)}} \begin{bmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{bmatrix}$$

2 POINTS

Problem 24.

A. Determine the Dirac spinors $v_1(p)$ and $v_2(p)$ for antiparticles.

B. Determine the polarization sum $\Lambda^-(p)$ for antiparticles.

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(24) B. Calculate the energy projection operator

$$\Lambda^-(\vec{p}) = -\sum_{r=1}^{2} v_r(\vec{p})\,\bar{v}_r(\vec{p})$$

$$= \frac{-1}{2m(E+mc^2)}\begin{pmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E/c + mc \end{pmatrix}\begin{pmatrix} p_x + ip_y & -p_z & 0 & -\left(\frac{E}{c}+mc\right)\end{pmatrix}$$

$$\frac{-1}{2m(E+mc^2)}\begin{pmatrix} p_z \\ p_x + ip_y \\ E/c + mc \\ 0 \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y & -\left(\frac{E}{c}+mc\right) & 0\end{pmatrix}$$

$$= \frac{-1}{2m(E+mc^2)}\begin{bmatrix} \vec{p}^{\,2} & 0 & -p_z\left(\frac{E}{c}+mc\right) & -(p_x - ip_y)\left(\frac{E}{c}+mc\right) \\ 0 & \vec{p}^{\,2} & -(p_x+ip_y)\left(\frac{E}{c}+mc\right) & +p_z\left(\frac{E}{c}+mc\right) \\ p_z(E/c+mc) & (p_x - ip_y)(E/c+mc) & -\left(\frac{E}{c}+mc\right)^2 & 0 \\ (p_x+ip_y)(E/c+mc) & -p_z(E/c+mc) & 0 & -\left(\frac{E}{c}+mc\right)^2 \end{bmatrix}$$

$$\frac{\vec{p}^{\,2}}{E+mc^2} = \frac{1}{c^2}\,\frac{E^2 - m^2c^4}{E+mc^2} = \frac{1}{c^2}(E - mc^2)$$

$$= \frac{-1}{2m}\begin{bmatrix} \frac{1}{c^2}(E-mc^2) & 0 & -p_z/c & -(p_x - ip_y)/c \\ 0 & \frac{1}{c^2}(E-mc^2) & -(p_x+ip_y)/c & p_z/c \\ p_z/c & (p_x-ip_y)/c & -\frac{1}{c}(E/c+mc) & 0 \\ (p_x+ip_y)/c & -p_z/c & 0 & -\frac{1}{c}\left(\frac{E}{c}+mc\right) \end{bmatrix}$$

$$= \frac{-1}{2mc}\left[\begin{array}{c|c} p^0 - mc & -\vec{\sigma}\cdot\vec{p} \\ \hline \vec{\sigma}\cdot\vec{p} & -p^0 - mc \end{array}\right] \Leftarrow 2\times 2 \text{ blocks}$$

$$= \frac{-1}{2mc}\left\{ p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - mc\right\} = \frac{-(\gamma\cdot p - mc)}{2mc}$$

$$= \frac{-(\not{p} - mc)}{2mc}$$

2 POINTS