Problem 5.1

Given $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$

where $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$.

$$\mathcal{L} = -\frac{1}{2} F_{\mu\nu}(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= \underbrace{-\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu)}_{\text{This is } \mathcal{L}_{Fermi}} + \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= \mathcal{L}_{Fermi} + \frac{1}{2}\partial_\mu\left[A_\nu(\partial^\nu A^\mu)\right] - \frac{1}{2}A_\nu(\partial^\nu\partial_\mu A^\mu) - \frac{1}{2}\partial_\mu\left[A^\mu(\partial_\nu A^\nu)\right] + \underbrace{\frac{1}{2}A^\mu(\partial_\mu\partial_\nu A^\nu)}_{\text{these cancel}}$$

$$= \mathcal{L}_{Fermi} + \partial_\mu\Big\{\underbrace{\frac{1}{2}A_\nu(\partial^\nu A^\mu) - \frac{1}{2}A^\mu(\partial_\nu A^\nu)}_{\text{call this } \Lambda^\mu(x).}\Big\}$$

Thus $\mathcal{L} = \mathcal{L}_{Fermi} + \partial_\mu \Lambda^\mu$.

Therefore $\mathcal{L}$ and $\mathcal{L}_{Fermi}$ are "equivalent".

(See Problem 2.1.)

The <u>actions</u> $\int \mathcal{L}\, d^4x$ and $\int \mathcal{L}_{Fermi}\, d^4x$ are equal.

Problem 5.2 Start with Eq. 5.28:

$$[a_r(\vec{k}), a_s^{+}(\vec{k}')] = \zeta_r \delta_{rs} \delta(\vec{k}, \vec{k}')$$

$$[a_r(\vec{k}), a_s(\vec{k}')] = [a_r^{+}(\vec{k}), a_s^{+}(\vec{k}')] = 0.$$

⦿ Calculate $[a_3(\vec{k}) - a_0(\vec{k}), a_3^{+}(\vec{k}') - a_0^{+}(\vec{k}')]$

$$= [a_3, a_3^{+}] - [a_0, a_3^{+}] - [a_3, a_0^{+}] + [a_0, a_0^{+}]$$

$$= +\delta(\vec{k}, \vec{k}') - 0 - 0 - \delta(\vec{k}, \vec{k}') = 0$$

⦿ The most general physical vacuum state is

$$|\Psi_{SL}\rangle = \sum_{\{n_1, n_2, n_3 \dots\}} c(n_1, n_2, n_3 \dots) \prod_{i=1}^{\infty} (a_3^{+}(\vec{k}_i) - a_0^{+}(\vec{k}_i))^{n_i} |0\rangle$$

PROOF $|\Psi_{SL}\rangle$ has no transverse photons: $a_{1,2}(\vec{k})|\Psi_{SL}\rangle = 0$.

AND $|\Psi_{SL}\rangle$ has $[a_3(\vec{k}) - a_0(\vec{k})]|\Psi_{SL}\rangle = 0$

because

$$[a_3(\vec{k}) - a_0(\vec{k})][a_3^{+}(\vec{k}) - a_0^{+}(\vec{k})]^n |0\rangle$$

$$= [a_3^{+}(\vec{k}) - a_0^{+}(\vec{k})]^n \underbrace{[a_3(\vec{k}) - a_0(\vec{k})]|0\rangle}_{=0}$$

$$= 0$$

Let $\alpha_i(\vec{k}) = a_3(\vec{k}) - a_0(\vec{k})$

$\alpha_i^{+}(\vec{k}) = a_3^{+}(\vec{k}) - a_0^{+}(\vec{k})$

- The norm of the state $|\Psi_{SL}\rangle$ is

$$\langle\Psi_{SL}|\Psi_{SL}\rangle = \sum_{\{n\}} c^*(\{n\}) \sum_{\{n'\}} c(\{n'\}) \underbrace{\langle 0| \prod_i (\alpha_i)^{n_i} \prod_j (\alpha_j^+)^{n'_j} |0\rangle}_{= \delta(n_i,0)\,\delta(n'_j,0)}$$

because $\alpha_i|0\rangle = 0$ and $\langle 0|\alpha_j^+ = 0$ and α_i commutes with α_j^+.

$$\langle\Psi_{SL}|\Psi_{SL}\rangle = |c(0,0,0,\ldots)|^2$$

- The most general state with a given set of <u>transverse</u> photons is

$$\prod_i [a_1^+(\vec{k}_i)]^{n_i^{(1)}} \prod_j [a_2^+(\vec{k}_j)]^{n_j^{(2)}} |\Psi_{SL}\rangle.$$

Problem 5.3 Calculate $\langle \psi_T' | A^\mu(x) | \psi_T' \rangle$

$$= \langle \psi_T | \{ 1 + \alpha^* (a_3 - a_0) \} A^\mu \{ 1 + \alpha (a_3^\dagger - a_0^\dagger) \} | \psi_T \rangle$$

$$= \langle \psi_T | A^\mu | \psi_T \rangle + \alpha^* \langle \psi_T | (a_3 - a_0) A^\mu | \psi_T \rangle$$
$$+ \alpha \langle \psi_T | A^\mu (a_3^\dagger - a_3^\dagger) | \psi_T \rangle + |\alpha|^2 \langle \psi_T | (a_3 - a_0) A^\mu (a_3^\dagger - a_0^\dagger) | \psi_T \rangle$$

$$= \langle A^\mu \rangle_{\psi_T} + \alpha^* \langle \psi_T | [a_3 - a_0 , A^\mu] | \psi_T \rangle$$
$$+ \alpha \langle \psi_T | [A^\mu, a_3^\dagger - a_0^\dagger] | \psi_T \rangle + |\alpha|^2 \langle \psi_T | \underbrace{[a_3 - a_0, A^\mu]}_{\text{This is a c-number}} (a_3^\dagger - a_0^\dagger) | \psi_T \rangle$$

because : $(a_3 - a_0) | \psi_T \rangle = 0$

and $\langle \psi_T | \, a_3^\dagger - a_0^\dagger = 0$

Also $\langle \psi_T | (\text{c-number})(a_3^\dagger - a_0^\dagger) = 0$

Now —

$$[a_3 - a_0 , A^\mu] = N_k e^{ik\cdot x} (\epsilon_3^\mu + \epsilon_0^\mu) \Longleftarrow \boxed{A^\mu = \sum_{\vec{k} r} N_k \epsilon^\mu(\vec{k}, r) \left(a(\vec{k}, r) e^{-ik\cdot x} + a^\dagger(\vec{k}, r) e^{+ik\cdot x} \right)}$$

Note : $\epsilon_3^\mu + \epsilon_0^\mu = \dfrac{k^\mu - (k\cdot n) n^\mu}{\sqrt{(k\cdot n)^2 - k^2}} + n^\mu$ $\quad \boxed{n^\mu = (1,0,0,0)}$

$\sqrt{(k\cdot n)^2 - k^2} = \sqrt{(k^0)^2 - k^2} = \sqrt{\vec{k}^2} = \omega$; also $k\cdot n = k^0 = \omega$

So $\epsilon_3^\mu + \epsilon_0^\mu = k^\mu / \omega$. $\quad \boxed{[a_3 - a_0 , A^\mu] = N_k e^{ik\cdot x} \frac{k^\mu}{\omega}}$

And — $[A^\mu, a_3^\dagger - a_0^\dagger] = N_k e^{-ik\cdot x} \dfrac{k^\mu}{\omega}$

$$\langle \psi_T' | A^\mu | \psi_T' \rangle = \langle A^\mu \rangle_{\psi_T} + \left(\alpha^* N_k e^{ik\cdot x} \frac{k^\mu}{\omega} + \alpha N_k e^{-ik\cdot x} \frac{k^\mu}{\omega} \right) \langle \Psi_T | \Psi_T \rangle$$

$= \langle \psi_T | A^\mu | \psi_T \rangle + \langle \psi_T | \partial^\mu \Lambda | \psi_T \rangle$ where $\boxed{\Lambda = \dfrac{N_k}{\omega} 2\mathrm{Re}\left(i\alpha e^{-ik\cdot x} \right)}$

Problem 5.4

Start with the Lagrangian density for the Klein Gordon equation, Eq. (3.22)

$$\mathcal{L}_0 = N\left(\phi^{+}_{,\alpha} \phi_{,}{}^{\alpha} - \mu^2 \phi^{+}\phi \right) \quad ; \quad \phi_{,\alpha} = \partial_\alpha \phi$$

normal ordering

Now replace $\partial_\alpha \phi \rightarrow D_\alpha \phi = \left(\partial_\alpha + \frac{ie}{\hbar c} A_\alpha \right) \phi$

and $\partial_\alpha \phi^{+} \rightarrow (D_\alpha \phi)^{+} = \left(\partial_\alpha - \frac{ie}{\hbar c} A_\alpha \right) \phi^{+}$

and set $\hbar = 1$ and $c = 1$; the new Lagrangian is

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ where

$$\mathcal{L}_I = N\left\{ (\partial_\alpha - ieA_\alpha)\phi^{+} (\partial^\alpha + ieA^\alpha)\phi - \mu^2 \phi^{+}\phi - \partial_\alpha \phi^{+} \partial^\alpha \phi + \mu^2 \phi^{+}\phi \right\}$$

$$= N\left\{ -ieA_\alpha \left[\phi^{+}\partial^\alpha \phi - (\partial^\alpha \phi^{+})\phi \right] + e^2 A_\alpha A^\alpha \phi^{+}\phi \right\}$$

$$= N\left\{ -A_\alpha s^\alpha + e^2 A_\alpha A^\alpha \phi^{+}\phi \right\}$$

where $s^\alpha = -ie\left[(\partial^\alpha \phi^{+})\phi - \phi^{+}(\partial^\alpha \phi) \right]$ = the current density.

The charge conjugation transformation C

Recall from chapter 3 : $\phi'(x) = C\phi(x)C^{+} = \eta_c \phi^{+}(x)$ $\quad |\eta_c|^2 = 1$

and $s'^{\alpha}(x) = -s^\alpha(x)$

So, $\mathcal{L}_I' = N\{-A_\alpha' s'^\alpha + e^2 A_\alpha' A'^\alpha \phi'^+ \phi'\}$

$$= N\{A_\alpha' s^\alpha + e^2 A_\alpha' A'^\alpha \phi^+ \phi\}$$

To make $\mathcal{L}_I' = \mathcal{L}_I$ we must have

$$A_\alpha' = C A_\alpha C^+ = -A_\alpha$$

Now consider a single photon state $|\vec{k}, r\rangle$.

$$C|\vec{k}, r\rangle = C \; a^+(\vec{k}, r)|0\rangle$$

$$= C a^+(\vec{k}, r) C^+ C |0\rangle \quad \text{because } C^+C = 1$$

$$= C a^+(\vec{k}, r) C^+ |0\rangle \quad \text{because } C|0\rangle = |0\rangle.$$

$$= -a^+(\vec{k}, r)|0\rangle \quad \text{because } C A^\mu C^+ = -A^\mu$$

$$= -|\vec{k}, r\rangle.$$

The photon has charge conjugation $= -1$.