The Theory of the Photon

Planck, Einstein $\longrightarrow$ the origins of quantum theory

Dirac $\longrightarrow$ the first quantum field theory

The equations of electromagnetism in empty space (i.e., no charged particles are present); using gaussian units,

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times \vec{E} = \frac{-1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$U = \frac{1}{2} \int (E^2 + B^2)\, d^3x$$

We can solve the field equations, by introducing potentials $\varphi(\mathbf{x},t)$ and $\mathbf{A}(\mathbf{x},t)$ …

$$\vec{E} = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

Then :

$$\nabla\cdot\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\vec{A}) - \nabla^2\phi = 0 \quad \underline{\text{is required}}$$

$$\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = \frac{-1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A}) + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A})$$
$$= 0 \quad \underline{\text{automatically}}$$

$$\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0 \quad \underline{\text{automatically}}$$

$$\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$
$$+ \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\phi) = 0$$

$\underline{\text{is required}}$

W.L.O.G. we can also require $\nabla \cdot \mathbf{A} = 0$,
which is called the *Coulomb gauge condition*.

Then $\Phi = 0$ and

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = 0 \quad \text{(wave equation)}$$

Thus, ***A*(*x***,*t*) *is an infinite set of harmonic oscillators.*

Or, I should say these are *coupled* oscillators;
different points ***x*** are coupled.

$$\ddot{q} + \omega^2 q = 0$$

$$\ddot{q}_i + \sum_j K_{ij} q_j = 0 \qquad (i=1 \ldots n)$$

$$i \longrightarrow \vec{x} \quad \text{and} \quad q_i \longrightarrow \vec{A}(\vec{x}).$$

And how do we calculate coupled oscillations?

We use the <u>*normal modes*</u>.

“Normal modes” of the electromagnetic field

The normal modes of the electromagnetic field vectors are transverse plane waves.

Try $\mathbf{A}(\mathbf{x},t) = C\,\boldsymbol{\varepsilon}\,e^{\,i(\mathbf{k}.\mathbf{x} - \omega t)}$

$\boldsymbol{\varepsilon}$ is a unit vector called the polarization vector

- The Coulomb gauge condition requires $\mathbf{k}\cdot\boldsymbol{\varepsilon} = 0$;

ie, the waves are transverse.

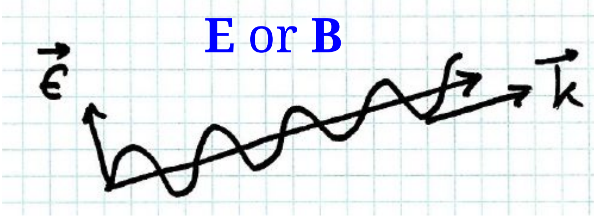
- The wave equation requires $\omega = c|\mathbf{k}|$, which is called the *dispersion relation*.

The wave velocity is

$$\frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k} = c$$

(This is the starting point for the theory of special relativity.)

“Normal modes”; or, better, plane wave solutions



$$\vec{\epsilon} \cdot \vec{k} = 0$$

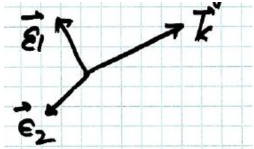
$$v_{phase} = \frac{\omega}{k} = c$$

The general solution of the field equation is a superposition of ~~normal modes~~ plane waves,

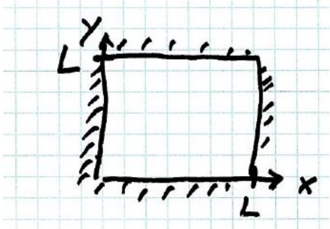
$$\vec{A}(\vec{x},t) = \sum_{\vec{k}} \sum_{\sigma=1}^{2} \hat{\epsilon}_{\sigma} \left[c(\vec{k},\sigma)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + c^{*}(\vec{k},\sigma)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \right]$$

where $\omega = ck$

and $\vec{k}$, $\vec{\epsilon}_1$, $\vec{\epsilon}_2$ form an orthogonal triad



Infinite space, or finite? For normalizability,
we often imagine a finite space,
with dimensions L x L x L,
and with periodic boundary conditions;
and finally take the limit L→ infinity .



$$\vec{A}(\vec{x} + L\hat{e}_i, t) = \vec{A}(\vec{x}, t)$$

periodic boundary conditions

Then for the plane waves,
***k** = (2π/L) **n** where **n** is integer valued.*

Volume $\Omega = L^3$.

Infinite volume limit : $\sum_{\mathbf{k}} \;\rightarrow\; \Omega \int d^3k \;/(2\pi)^3$

So far, I have only discussed the wave solutions of the field equations. *Where are the photons?*

Following Dirac, we replace the c-numbers ,

$c(\mathbf{k},\sigma)$ and $c^*(\mathbf{k},\sigma)$,

by annihilation and creation operators,

$c(\mathbf{k},\sigma) \rightarrow a_{\mathbf{k}\sigma}$ and $c^*(\mathbf{k},\sigma) \rightarrow a^\dagger_{\mathbf{k}\sigma}$

with

$$[\ a_{\mathbf{k}\sigma}\ ,\ a^\dagger_{\mathbf{k}\sigma}\ ] = 1$$

$$[\ a_{\mathbf{k}\sigma}\ ,\ a^\dagger_{\mathbf{k}'\sigma'}\ ] = 0 \quad \text{for } \{\mathbf{k}',\sigma'\} \neq \{\mathbf{k},\sigma\}$$

i.e., $$[\ \mathbf{a}_{\mathbf{k}\sigma}\ ,\ \mathbf{a}^\dagger_{\mathbf{k}'\sigma'}\ ] = \delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\sigma\sigma'}$$

All other commutators are 0 ; e.g., $[a_{\mathbf{k}\sigma}\ ,\ a_{\mathbf{k}'\sigma'}] = 0$.

So we write

$$\mathbf{A}(\mathbf{x},t) = \sum N_k \,\boldsymbol{\varepsilon}_{\mathbf{k}\boldsymbol{\sigma}} \{ a_{\mathbf{k}\boldsymbol{\sigma}} \; e^{i(\mathbf{k}.\mathbf{x} - \omega t)} + a^{\dagger}_{\mathbf{k}\boldsymbol{\sigma}} \, e^{-i(\mathbf{k}.\mathbf{x} - \omega t)} \}$$

and that is the quantized electromagnetic field,
in the Heisenberg picture. (Mandl and Shaw, 1.38)

This is supposed to be a quantum theory. Then there must be a Hamiltonian (= the generator of translation in time).

What is the Hamiltonian? *(Homework Problem 3)*

$$H = \sum \hbar\omega \, a^{\dagger}_{k\sigma} a_{k\sigma} + \text{const.}$$

<u>The normalization factor</u> N_k

$$N_k = \text{Sqrt}[\, \hbar \, c^2 \; / \, (2\Omega\omega) \,]$$ (remember, $\omega = ck$)

which is necessary so that $H = 1/2 \int (E^2 + B^2) \, d^3x$.

Pictures *Here is something that we need for quantum field theory. You have already studied it, but we'll go over it again.*

- All predictions in quantum theory are based on matrix elements; e.g., $O_{\alpha\beta} = \langle \alpha \mid O \mid \beta \rangle$.
- Matrix elements may depend on time. But what depends on time—the states or the observables?
- In the Schroedinger picture, the states depend on time, but the observables do not depend on time.
- In the Heisenberg picture, the states do not depend on time, but the observables depend on time.
- *Essential:* $\langle \alpha,t \mid O \mid \beta,t \rangle_{\text{Schr.}} = \langle \alpha \mid O(t) \mid \beta \rangle_{\text{Heis.}}$

(Homework Problem 4)

(Needed for Homework Problem 4)

Time dependence in the Schroedinger picture

$$i \hbar (\partial/\partial t) \; | \alpha, t \rangle = H \, | \alpha, t \rangle;$$

or,

$$| \alpha, t \rangle = \exp(- i H t / \hbar) \, | \alpha, 0 \rangle .$$

Time dependence in the Heisenberg picture

$$\exp(+ i H t / \hbar) \, \mathbf{O} (0) \exp(- i H t / \hbar) = \mathbf{O} (t) \quad ;$$

or,

$$- i \hbar (\partial/\partial t) \, \mathbf{O} (t) = [\; H , \mathbf{O} (t)] .$$

Homework due Friday, January 20 …

Problem 3. The quantum field for the free electromagnetic field $\mathbf{A}(\mathbf{x},t)$ *in the Heisenberg picture* is written as an expansion in plane waves, with the annihilation and creation operators, $a_{\mathbf{k}\sigma}$ and $a^\dagger_{\mathbf{k}\sigma}$.

(a) Let $H = \sum \hbar\omega\, a^\dagger_{k\sigma} a_{k\sigma}$ + const. Show that $\mathbf{A}(\mathbf{x},t)$ obeys the Heisenberg equation of motion.

(b) Show that $H = 1/2 \int (E^2 + B^2)\, d^3x$.

Problem 4.

Start with the equations for time dependence in the Schroedinger and Heisenberg pictures.

Prove $\langle \alpha,t \mid O \mid \beta,t \rangle_{\text{Schr.}} = \langle \alpha \mid O(t) \mid \beta \rangle_{\text{Heis.}}$