Summarize the theory of free photons

In the Heisenberg picture,

$$\mathbf{A}(\mathbf{x},t) = \sum N_k \, \boldsymbol{\varepsilon}_{\mathbf{k\sigma}} \{ \, a_{k\sigma} \, e^{\,i\,(\mathbf{k}.\mathbf{x} - \omega t)} + a^{\dagger}_{k\sigma} \, e^{-i\,(\mathbf{k}.\mathbf{x} - \omega t)} \, \}$$

$$N_k = \mathrm{Sqrt}[\ \hbar\, c^2 \ / \ (2\omega\Omega)\ ]$$

$$H_{rad} = \sum \hbar\omega \ a^{\dagger}_{k\sigma} \ a_{k\sigma}$$ (& remember, $\omega = ck$)

$$[\, a_{\mathbf{k\sigma}} \ , \ a^{\dagger}_{\mathbf{k'\sigma'}} \,] = \delta_{\mathbf{k,k'}} \ \delta_{\boldsymbol{\sigma\sigma'}}$$ *(all other commutators are 0)*

So , $a_{\mathbf{k\sigma}}$ annihilates a photon of the mode { **k**, $\boldsymbol{\varepsilon}_{\mathbf{k\sigma}}$ };

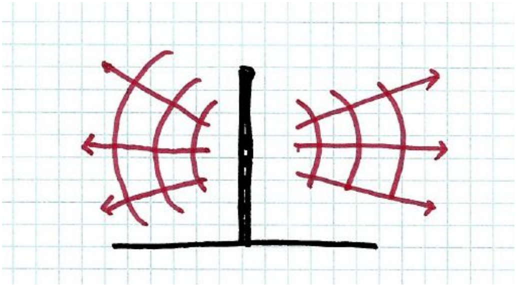
and $a^{\dagger}_{\mathbf{k\sigma}}$ creates a photon.

The photon field **A**(**x**,t) creates and annihilates photons.

Coherent states and the classical limit of the electromagnetic field (Glauber)

We know that Maxwell's equations describe macroscopic electromagnetic waves.

For example, think of radio waves.



How is that related to photons?

Consider just a single mode of oscillation, with quantum numbers {**k**,σ}.

The *general* state vector for the e.m. field is

$$| \Psi \rangle = \sum_{n=0}^{\infty} c_n \, | n \rangle$$ *[Heisenberg picture]*

where $| n \rangle = 1/\mathrm{Sqrt}[n!] \; (a^\dagger)^n \; | 0 \rangle$.

The *coherent* state is

$$c_n = \exp(-\alpha^2/2) \; \alpha^n \; / \, \mathrm{Sqrt}[n!] \quad .$$

What are the electric and magnetic fields?

Calculate the *expectation value* of the electric field **operator** in the coherent state.

$$\langle \Psi \mid \mathbf{E(x,t)} \mid \Psi \rangle \;;$$

$$\mid \Psi \rangle = \sum c_n \mid n \rangle \;;$$

$$c_n = \exp(-\alpha^2/2)\ \alpha^n \,/\, \mathrm{Sqrt}[n!]\ .$$

This is what we mean by the classical electric field.

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \sum_k N_k \hat{\epsilon}_k \left\{ a_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} - a^{\dagger}_{\vec{k}}\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \right\} \frac{i\omega}{c}$$

$$\langle n' | a | n \rangle = \sqrt{n}\ \delta(n', n-1)$$

$$\langle n' | a^{\dagger} | n \rangle = \sqrt{n+1}\ \delta(n', n+1)$$

$$\langle \vec{E} \rangle = N \hat{\varepsilon} \sum_{n' n} c^{*}_{n'}\, c_n \left\{ \sqrt{n}\ \delta(n', n-1)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} - \sqrt{n+1}\ \delta(n', n+1)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \right\} \frac{i\omega}{c}$$

$$= \frac{i\omega}{c} N \hat{\varepsilon} \sum_n \left\{ \sqrt{n}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, e^{-\alpha^2} \alpha^{2n-1} / \sqrt{n!}\sqrt{(n-1)!} - \sqrt{n+1}\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, e^{-\alpha^2} \alpha^{2n+1} / \sqrt{n!}\sqrt{(n+1)!} \right\}$$

$$= \frac{i\omega}{c} N \hat{\varepsilon}\, e^{-\alpha^2} \left\{ \sum_n \frac{\alpha^{2n-1}}{(n-1)!}\, e^{i(k\cdot x-\omega t)} - \sum_n \frac{\alpha^{2n+1}}{n!}\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \right\}$$

$$= \frac{i\omega}{c} N \hat{\varepsilon}\, \alpha \left\{ e^{i\Theta} - e^{-i\Theta} \right\}$$

$$= -\frac{2\omega}{c} N \hat{\varepsilon}\, \alpha\, \sin(\vec{k}\cdot\vec{x}-\omega t)$$

There is also a classical magnetic field.

$\langle \Psi \mid \mathbf{B(x,t)} \mid \Psi \rangle$

$$\langle \Psi | \vec{B} | \Psi \rangle = -2N(\vec{k} \times \vec{\epsilon})\alpha \sin(\vec{k} \cdot \vec{x} - \omega t)$$

These functions, <***E(x***,*t)*> and <***B(x***,*t)*> , are plane wave solutions of Maxwell's equations.

These are what we mean by a classical electromagnetic wave.

How many photons are there in the classical wave?

The number of photons in the coherent state is ***indeterminate.***

$$\langle n \rangle = \langle \Psi | \; a^\dagger a \; | \Psi \rangle = \alpha^2 \; .$$

(Note that $< H > = \hbar\omega \, \alpha^2 = \hbar\omega < n >$)

$$\langle n^2 \rangle = \langle \Psi | \; a^\dagger a \, a^\dagger a \, | \Psi \rangle$$
$$= \langle \Psi | \; a^\dagger \, (1 + a^\dagger a) \, a \, | \Psi \rangle$$
$$= \alpha^2 + \alpha^4 \; .$$

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \alpha^2 = \langle n \rangle$$

$$\therefore \; \Delta n = \sqrt{\langle n \rangle}$$

⬅ *The uncertainty in the number of photons.*

Homework Problem 6.

$$a \, |\Psi\rangle = \alpha \, |\Psi\rangle$$

Is light a wave or a stream of particles?

- ❑ Does the photon theory mean that light is a stream of particles?

No, because the number of photons in a macroscopic electromagnetic wave is *indeterminate*.

- ❑ Does the photon theory mean that Maxwell’s electromagnetic wave theory is obsolete?

No, because the expectation values of the quantum fields are a Maxwellian wave. If the number of photons is large, the quantum effects are negligible.

- ❑ So what is a photon?

It’s the *field quantum* — the smallest energy excitation of the electromagnetic field.

Next topic :

Interactions of Radiation and Matter

Homework problems 1 -- 6

are due next Friday, January 20.

Additional Homework problems due Friday, January 20 …

Problem 5.

WKAR-FM is a public radio station in East Lansing, Michigan; broadcasting on the FM dial at 90.5 MHz. It is owned by Michigan State University, and is sister station to the AM radio and television stations with the same call letters. The station signed on for the first time on October 4, 1948 as the Lansing area's first FM station.

The station's 85,000-watt signal, combined with a 269.3 meter antenna can be heard as far east as Flint and the Detroit suburbs, and as far west as Grand Rapids and Kalamazoo. WKAR-FM is a "Superpower Grandfathered" Class B FM station, providing a signal 7.6 db stronger than would be granted today under current U.S. Federal Communications Commission (FCC) rules.

(A) Estimate the mean number of photons emitted per second.

(B) Estimate the RMS variation of the number of photons emitted per second.

Problem 6.

Show that the photon coherent state is an eigenstate of the annihilation operator.