Chapter 1

The Theory of the Photon

Sections 1.1 – 1.2

- quantum numbers { $\mathbf{k}, \sigma$ }
- operators $a_{\mathbf{k}\boldsymbol{\sigma}}$ and $a^{\dagger}_{\mathbf{k}\boldsymbol{\sigma}}$
- $H_{rad} = \Sigma\, \hbar\omega\, a^{\dagger}_{k\sigma}\, a_{k\sigma}$
- $\mathbf{P}_{rad} = \Sigma\, \hbar\mathbf{k}\, a^{\dagger}_{k\sigma}\, a_{k\sigma}$
- Field operators $\mathbf{A}(\mathbf{x},t)$, $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$

Problem 1-1 (pages 24-25)

- The coherent state (Glauber)

Sections 1.3 – 1.4

- Interactions of radiation with matter

Interactions of Radiation and Matter

$$H = H_{atom} + H_{rad} + H_{int}$$

We know

$$H_{atom} = \frac{p^2}{2m} - \frac{Ze^2}{4\pi r} \quad \text{(one electron atom or ion)}$$

$$H_{rad} = \sum_{\vec{k}\sigma} \hbar\omega \, a^{+}_{\vec{k}\sigma} a_{\vec{k}\sigma} \quad \text{where } \omega = ck.$$

Now, what is H_{int}?

Start with the Hamiltonian for an electron in an electromagnetic field,

$$H = \frac{(\vec{p} + e/c\,\vec{A})^2}{2m} - e\phi$$

(electron charge = – e)

Why?

Check Hamilton’s equations,

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = -\frac{1}{m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)_j \frac{e}{c}\frac{\partial A_j}{\partial \vec{x}} + e\frac{\partial \phi}{\partial \vec{x}}$$

$$= -\dot{x}_j \frac{e}{c}\frac{\partial A_j}{\partial \vec{x}} + e\nabla\phi \qquad (1)$$

Also,

$$\dot{\vec{p}} = m\ddot{\vec{x}} - \frac{e}{c}\dot{\vec{A}}$$

$$= \vec{F} - \frac{e}{c}\left[\frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}\right] \qquad (2)$$

$$\therefore \quad \vec{F} = -e\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$ as required

$$v\times\vec{B} = \vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}$$

Now write H = $p^2/2m - e\,\varphi(\mathbf{x}) + H_{int}$

So,

$$H_{int} = \frac{e}{mc}\vec{A}(\vec{x})\cdot\vec{p} + \frac{e}{2mc^2}A^2(\vec{x})$$

Recall,

$$\vec{A}(\vec{x}) = \sum_{\vec{k}\sigma} N_k \vec{\varepsilon}_{k\sigma}\left[a_{\vec{k}\sigma}\, e^{i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}\sigma}\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

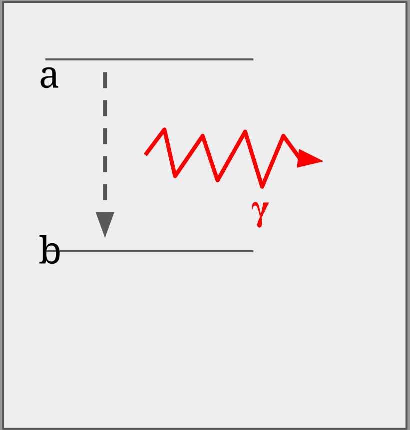
(in the Schroedinger picture !)

The first term of H_{int} will annihilate or create a photon.
The second term could annihilate or create 2 photons.

N_k = SQRT[$\hbar c^2$ / $(2\omega\Omega)$]

Radiative decays in atomic physics

$a \longrightarrow b + \gamma$ where $\varepsilon_a > \varepsilon_b$



We'll calculate the transition rate by Fermi's Golden Rule, from perturbation theory, treating H_{int} as a perturbation.

$$\text{Rate} = \frac{2\pi}{\hbar} \sum_f \left| \langle f | H_{int} | i \rangle \right|^2 \delta(E_f - E_i)$$

(first order approximation in time-dependent perturbation theory).

The initial state is $|i\rangle = |a\rangle_{\text{atom}} \, |\text{vacuum}\rangle_{\text{rad.}}$

The final state is $|f\rangle = |b\rangle_{\text{atom}} \, |\mathbf{k}, \boldsymbol{\varepsilon}\rangle_{\text{rad.}}$

i.e., $a^\dagger_{\mathbf{k}\boldsymbol{\varepsilon}} \, |\text{vacuum}\rangle_{\text{rad.}}$

Calculate the transition matrix element

$\langle f \mid H_{int} \mid i \rangle =$

$$\langle b | \langle \vec{k}\vec{\varepsilon} | \frac{e}{mc} \vec{A}\cdot\vec{p} \,|\, \text{vacuum} \rangle \,|\, a \rangle$$

$$= \frac{e}{mc} \langle b | \underbrace{\langle \vec{k}\vec{\varepsilon} | \vec{A} | \text{vac.} \rangle}_{= N_{\vec{k}}\, \hat{\varepsilon}_{\vec{k}\sigma}\, e^{-i\vec{k}\cdot\vec{x}}} \cdot \vec{p} \,|\, a \rangle$$

(= coefficient of $a^{\dagger}_{\vec{k}\sigma}$ in $\vec{A}(\vec{x})$)

$$= \frac{e}{mc} N \hat{\varepsilon} \cdot \langle b | e^{-i\vec{k}\cdot\vec{x}} \vec{p} \,|\, a \rangle$$

The size of the atom is << the wavelength of the photon ,

10^{-10} m << 100 nm = 10^{-7} m ;

so we can approximate $e^{-i\,\mathbf{k}.\mathbf{x}} \approx 1$. *(electric dipole approx.)*

$$\text{Rate} = \frac{2\pi}{\hbar} \sum_{\vec{k}\sigma} \left(\frac{e}{mc}\right)^2 N^2 \left| \hat{\varepsilon} \cdot \langle b | \vec{p} | a \rangle \right|^2 \delta(\epsilon_b + \hbar\omega - \epsilon_a)$$

The rest in just math.

Example:

The decay H(2p) → H(1s) + γ for atomic hydrogen.

The atomic matrix element is

$$\langle 2p, m \mid \mathbf{p} \mid 1s\rangle \qquad \text{for } m \in \{ 0, +1, -1 \}$$

This is just a problem in ordinary quantum mechanics.

The calculation is not difficult.

Recall the wave functions,

$$\Phi_{1s}(x) = \exp(-r/a_0)\ /(\pi\, a_0^{\,3})^{1/2}$$

& let's take m=0 ,

(the total decay rate would be the same for m = – 1 or +1)

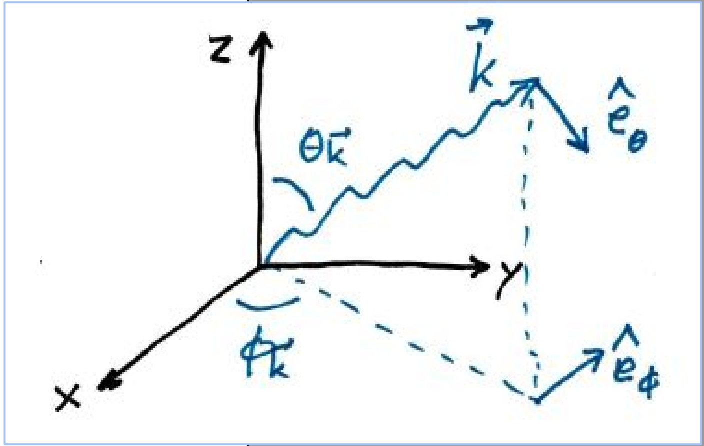
$$\Phi_{2p}(x) = (r/a_0\ )\ \ \exp(-r\,/2a_0)\ \cos\theta\ /(\,32\,\pi\, a_0^{\,3})^{1/2}$$

$$\langle 2p, 0 \mid \mathbf{p} \mid 1s\rangle = -i\hbar \int r^2\, dr\, \sin\theta\, d\theta\, d\varphi\ \ \Phi^*_{2p}\, \nabla \Phi_{1s}$$

$$= \mathbf{e}_z\, (i\hbar/a_0)\, (2/3)^4\, \sqrt{2}$$

Sum over polarizations

To get the *unpolarized decay rate* we must sum over the final two polarizations , $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$. These are orthogonal and perpendicular to $\mathbf{k}$. We can let $\boldsymbol{\varepsilon}_1 = \mathbf{e}_\varphi$ and $\boldsymbol{\varepsilon}_2 = \mathbf{e}_\theta$.
Note that $\mathbf{e}_z \bullet \mathbf{e}_\varphi = \mathbf{0}$ and $\mathbf{e}_z \bullet \mathbf{e}_\theta = -\sin\theta_k$



So,

$$\sum_\sigma \left| \boldsymbol{\varepsilon}_\sigma \bullet \langle b|\, \mathbf{p}\, |a\rangle \right|^2 = (\mathbf{e}_\theta \bullet \mathbf{e}_z)^2\, (\hbar/a_0)^2\, (2^9 / 3^8)$$

$$= \sin^2\theta_k \, (\hbar/a_0)^2\, (2^9 / 3^8)$$

Unpol. decay rate =

$$\frac{2\pi}{\hbar} \sum_{\vec{k}} \left(\frac{e}{mc}\right)^2 N^2 \sin^2\theta_k \left(\frac{\hbar}{a_0}\right)^2 \frac{2^9}{3^8}$$

$$\delta(\epsilon_b + \hbar\omega - \epsilon_a)$$

The differential decay rate (for m = 0)

We need to calculate the sum over **k.**

Remember the infinite volume limit,

$$\sum_{\vec{k}} = \frac{\Omega}{(2\pi)^3}\int d^3k$$

(periodic boundary conditions)

and $d^3k = k^2 dk\, d\Omega_k$

To get the *total rate*, integrate over all directions of **k**; to get the *differential rate*, don't integrate over directions…

$$\frac{dR}{d\Omega_k} = \left\{\frac{2\pi}{\hbar}\frac{\Omega}{(2\pi)^3}\left(\frac{e}{mc}\right)^2 \cancel{N^2}\left(\frac{\hbar}{a_0}\right)^2 \frac{2^9}{3^8}\right\}\left[\sin^2\theta_k\right] \frac{\hbar c^2}{2\omega\Omega}$$

$$\times \int k^2 dk\ \delta(\epsilon_b + \hbar c k - \epsilon_a)$$

important cancellation $\Omega/\Omega = 1$; ang. distribution $\propto \sin^2\theta_k$ because $\Delta m = 0$

and $\displaystyle\int \frac{k^2 dk}{\omega}\,\delta(\epsilon_b + \hbar c k - \epsilon_a) = \frac{1}{\hbar c^2}\left(\frac{\epsilon_b - \epsilon_a}{\hbar c}\right)$

$$\frac{dR}{d\Omega_k} = \left\{ \frac{2\pi}{\hbar} \frac{\Omega}{(2\pi)^3} \left(\frac{e}{mc}\right)^2 \cancel{N^2} \left(\frac{\hbar}{a_0}\right)^2 \frac{2^9}{3^8} \right\} \left[\sin^2\theta_k\right] \frac{\hbar c^2}{2\omega\Omega}$$

$$\times \int k^2 dk \; \delta(\epsilon_b + \hbar c k - \epsilon_a)$$

The k integration is evaluated with the delta function.

!!! Remember how to do delta function integrations:

$$\int_{-\infty}^{\infty} \delta(ax)\; f(x)\; dx = 1/a \quad f(0)$$

(i.e., needs the Jacobian factor)

$$\int \frac{k^2 dk}{\omega} \delta(\epsilon_b + \hbar c k - \epsilon_a) = \frac{1}{\hbar c^2}\left(\frac{\epsilon_b - \epsilon_a}{\hbar c}\right)$$

A general formula is given in Mandl & Shaw:

Equation 1.50

$$\frac{dR}{d\Omega_k} = \frac{e^2}{8\pi^2} \frac{\omega^3}{\hbar c^3} |\langle b | \vec{x} | a \rangle|^2 \sin^2\theta_k$$

in rationalized cgs units;

$$\alpha = \frac{e^2}{4\pi\hbar c} = \frac{1}{137}.$$

E.g., estimate

Order of magnitude for $H_{2p} \to H_{1s} + \gamma$,

$$\frac{\alpha}{c^2} \cdot \frac{1}{\hbar^3}\underbrace{\left(\frac{me^4}{2\hbar^2}\right)^3}_{\text{Rydberg en.}} \underbrace{\left(\frac{\hbar^2}{me^2}\right)^2}_{\text{Bohr rad.}} \propto m\alpha^5$$

Homework problems due Friday, January 27

Hint: For these calculations start with Equation (1.50).

Problem 7. Calculate accurately the mean lifetime of the 2p state of atomic hydrogen.

Problem 8. Estimate the mean lifetime for a nuclear gamma decay process. (You only need to calculate the order of magnitude.)