Mandl and Shaw reading assignments

Chapter 2 Lagrangian Field Theory
2.1 Relativistic notation ✔
2.2 Classical Lagrangian field theory ✔
2.3 Quantized Lagrangian field theory
2.4 Symmetries and conservation laws
Problems; 2.1 2.2 2.3 2.4 2.5

Chapter 3 The Klein-Gordon Field
3.1 The real Klein-Gordon field
3.2 The complex Klein-Gordon field
3.3 Covariant commutation relations
3.4 The meson propagator
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LAGRANGIAN FIELD THEORY AND CANONICAL QUANTIZATION (CHAPTER 2)

In the history of science, the first field theory was electromagnetism. (Maxwell)

There are 2 vector fields, $E$ and $B$.

In spacetime we have a field tensor. (Minkowski)

- The classical field theory describes electromagnetic waves with $\omega = ck$.
- The quantum field theory describes photons. (Chapter 1)
- We can derive the theory from a Lagrangian, and then quantize it. But there are some subtleties, due to gauge invariance! (Chapter 5)

Electromagnetism isn’t very interesting without sources, i.e., charges.

- Add the electron field (Chapter 4) which leads to Quantum Electrodynamics. (Chapter 7)
Recall the example of the Schrödinger equation, from Monday.

- **Classical field theory:** \( \psi(x,t) \) is a complex function.
- **Quantum field theory:** \( \psi(x,t) \) is a non-hermitian operator.

Now another example: (Read SECTIONs 2.1, 2.2, 3.1)

**A REAL SCALAR FIELD** \( \phi(x,t) \)

This example is relativistically covariant.

\[
\begin{align*}
\mathcal{L} & = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{c^2}{2} \left( \nabla \phi \right)^2 - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \phi^2 \\
A & = \int \left\{ \frac{1}{2} \phi^2 - \frac{c^2}{2} \nabla \phi \cdot \nabla \phi - \frac{1}{2} \left( \frac{mc^2}{\hbar} \right)^2 \phi^2 \right\} d^3x \, dt \\
\delta A & = \int \delta \phi \left\{ \phi \, \delta \phi - c^2 \nabla \phi \cdot \nabla (\delta \phi) - \left( \frac{mc^2}{\hbar} \right)^2 \phi \, \delta \phi \right\} d^3x \, dt \\
& = \int \delta \phi \left\{ -\phi + c^2 \nabla^2 \phi - \left( \frac{mc^2}{\hbar} \right)^2 \phi \right\} d^3x \, dt \\
& = 0 \text{ for any variation } \delta \phi(x,t).
\end{align*}
\]

\[
\begin{align*}
[i \hbar \gamma^0 - \gamma^i \nabla_i + (\gamma^0 \frac{mc}{\hbar})^2] \phi & = 0 \\
& \text{the Klein–Gordon equation}
\end{align*}
\]
We can solve the Klein-Gordon equation, in plane waves...

$$ \dddot{\phi} - c^2 \nabla^2 \phi + \left( \frac{mc^2}{\hbar} \right)^2 \phi = 0 $$

The Klein-Gordon equation

$$ \phi(\vec{x}, t) = C e^{i \left( \vec{k} \cdot \vec{x} - \omega t \right)} $$

where

$$ -\omega^2 + c^2 k^2 + \left( \frac{mc^2}{\hbar} \right)^2 = 0 $$

$$ \omega = \pm \sqrt{c^2 k^2 + m^2 c^4 / \hbar^2} $$

For example,

$$ \hbar \omega = \pm \sqrt{(c^2 k)^2 + (mc^2)^2} $$

Note that this is the energy ($\hbar \omega$) and momentum ($\hbar \vec{k}$) relation of special relativity.

\( What \ are \ the \ negative \ energy \ solutions? \)

The general solution (**Hermitian**) is

$$ \phi(\vec{x}, t) = \sum_k N_k \left\{ a_k \ e^{i (\vec{k} \cdot \vec{x} - \omega t)} + a_k^\dagger \ e^{-i (\vec{k} \cdot \vec{x} - \omega t)} \right\} $$

**Quantization**

We may anticipate

$$ [ a_k, a_{k'}^\dagger ] = \delta_{k k'} \ (k, k') $$

$$ [ a_k, a_k ] = 0 \ \text{and} \ [ a_k^\dagger, a_k^\dagger ] = 0 $$

We'll derive this from Dirac’s **canonical quantization**. Recall,

$$ [ q, p ] = \hbar \text{ } where \ \ \ p = \partial L / \partial \dot{q} $$

$$ a_k^\dagger \text{ in the quantized theory}$$
The Hamiltonian

\[ H = p \dot{q} - L, \]
rewritten in terms of \( q \) and \( p \)

\[ H = \int \left( \pi(x) \dot{\varphi}(x) - \mathcal{L} \right) d^3x \]
rewritten in terms of \( \varphi(x) \) and \( \pi(x) \)

**Homework problem 18.**

(A) Write \( H \) in terms of \( \pi(x) \) and \( \varphi(x) \).

(B) Write \( H \) in terms of \( a_k \) and \( a_k^\dagger \).

**Homework problem 19.**

Determine the Feynman propagator for the free scalar field:

\[ \Delta_F(x-y) = \langle 0 \mid T \varphi(x) \varphi(y) \mid 0 \rangle. \]

Here \( x \) stands for the 4-vector spacetime coordinate, \( x^\mu = (x^0, x, y, z) \).
Next: A real scalar field $\phi$ with a source $\rho$.

To make it simpler, set $\hbar = 1$ and $c = 1$ ("natural units"). At the end of a calculation we can restore the factors of $\hbar$ and $c$ by dimensional analysis (i.e., simple units analysis).

The field equation is a linear inhomogeneous equation;

so $\phi(x,t) = \phi_{\text{particular}}(x,t) + \phi_{\text{homogeneous}}(x,t)$.

The \textit{particular solution} comes from the source; e.g., it could be a mean field produced by a static source; or, waves radiated by a time dependent source. The \textit{homogeneous solution} consists of harmonic waves.

\begin{equation}
-\nabla^2 \phi_0 + m^2 \phi_0 = \rho(x)
\end{equation}

We need the Green's function $G$:

\begin{equation}
-\nabla^2 + m^2 \quad \text{i.e.,}
\end{equation}

\begin{equation}
(-\nabla^2 + m^2) G(x-y) = \delta^3(x-y)
\end{equation}

Then

\begin{equation}
\phi_0(x) = \int G(x-y) \rho(y) \, d^3y
\end{equation}
The Green’s function of $-\nabla^2 + m^2$

$(-\nabla^2+\mu^2)\,G(\xi) = \delta^3(\xi)$

$G(\xi) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\xi \cdot \mathbf{k}}}{k^2+m^2}$

$= \frac{e^{-m|\xi|}}{4\pi |\xi|} \quad (\text{with } \hbar=1 \text{ and } c=1)$

$= \frac{1}{4\pi |\xi|}$

Example

Suppose $\rho(x) = \rho_0 \theta(\ a - r ).$
An interaction Lagrangian density

\[ \mathcal{L}_{\text{interaction}} = g \, \Psi \dagger \Psi \phi \]

- This \( \mathcal{L}_{\text{int}} \) acts as a source for \( \phi \), with
  \[ \rho(x,t) = g \, \Psi \dagger \Psi . \]

- It also acts as a potential for \( \Psi \):
  \[ V_{\text{int}}(x,t) = -g \, \phi(x,t) . \]

\[ \Rightarrow \] The field equations;
   i.e., Lagrange’s equations,

\[-\frac{\hbar^2}{2m} \nabla^2 \Psi - g \phi \Psi = i\hbar \frac{\partial \Psi}{\partial t} \]
\[ \frac{\hbar^2 \phi}{2\epsilon^2} - \nabla^2 \phi + m^2 \phi = g \Psi^\dagger \Psi \]

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Homework due Fri Feb 10

Problem 18.
For the free real scalar field,
(A) Write \( H \) in terms of \( \pi(x) \) and \( \phi(x) \).
(B) Write \( H \) in terms of \( a_k \) and \( a_k^\dagger \).

Problem 19.
(A) Mandl and Shaw problem 3.3.
(B) Mandl and Shaw problem 3.4.

Problem 20.
The Yukawa theory problem.