Mandl and Shaw reading assignments

Chapter 2 Lagrangian Field Theory

2.1 Relativistic notation ✔

2.2 Classical Lagrangian field theory ✔

2.3 Quantized Lagrangian field theory

2.4 Symmetries and conservation laws

Problems; 2.1 2.2 2.3 2.4 2.5

Chapter 3 The Klein-Gordon Field

3.1 The real Klein-Gordon field

3.2 The complex Klein-Gordon field

3.3 Covariant commutation relations

3.4 The meson propagator

Problems; 3.1 3.2 3.3 3.4 3.5

LAGRANGIAN FIELD THEORY AND CANONICAL QUANTIZATION **(CHAPTER 2)**

In the history of science, the first field theory was electromagnetism. *(Maxwell)*

There are 2 vector fields, **E** and **B**.

In spacetime we have a field tensor. *(Minkowski)*

$$
F^{\mu\nu}(x) = \begin{array}{c|cccc}
\mu \backslash \nu & 0 & 1 & 2 & 3 \\
\hline
0 & & & & \\
1 & & & & \\
2 & & & & \\
3 & & & &
\end{array}
\qquad
\begin{array}{c}
0 \\ 1 \\ 2 \\ 3
\end{array}
\begin{bmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_x \\
-E_z & B_y & -B_x & 0
\end{bmatrix}
$$

$x = (x^0, \vec{x})$

Or, $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$

$A^\mu = (A^0(x), \vec{A}(x))$ and $\partial^\mu = g^{\mu\nu} \dfrac{\partial}{\partial x^\nu}$

- The *classical field theory* describes electromagnetic waves with $\omega = ck$.
- The *quantum field theory* describes photons. (Chapter 1)
- We can derive the theory from a Lagrangian, and then quantize it. *But there are some subtleties, due to gauge invariance! (Chapter 5)*

Electromagnetism isn't very interesting without sources , i.e., *charges*.

- Add the electron field (Chapter 4) which leads to **Q**uantum **E**lectro**D**ynamics. (Chapter 7) .

Recall the example of the Schroedinger equation, from Monday.

- *Classical field theory*:

$\psi(\mathbf{x},t)$ is a complex function.

- *Quantum field theory:* $\psi(\mathbf{x},t)$ is a non-hermitian operator.

$$\text{Action} = \int_{t_1}^{t_2} dt \int d^3x \left\{ \frac{-i\hbar}{2}\left(\frac{\partial \psi^*}{\partial t}\psi - \psi^* \frac{\partial \psi}{\partial t} \right) - \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi - V\psi^*\psi \right\}$$

$$\Rightarrow \quad i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$[\psi(x), \psi^+(x')]_{\mp} = \delta^3(x-x')$$

$$[\psi(x), \psi(x')]_{\mp} = 0$$

Now another example:

(Read SECTIONs 2.1, 2.2. 3.1)

<u>A REAL SCALAR FIELD</u> $\varphi = \varphi(\mathbf{x},t)$

This example is relativistically covariant.

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{c^2}{2}(\nabla\phi)^2 - \frac{1}{2}\left(\frac{mc^2}{\hbar}\right)^2 \phi^2$$

$$A = \int \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2}(\nabla\phi)^2 - \frac{1}{2}\left(\frac{mc^2}{\hbar}\right)^2\phi^2 \right\} d^3x\, dt$$

$$\delta A = \int \left\{ \dot{\phi}\,\delta\dot{\phi} - c^2\,\nabla\phi\cdot\nabla(\delta\phi) - \left(\frac{mc^2}{\hbar}\right)^2 \phi\,\delta\phi \right\} d^3x\, dt$$

$$= \int \delta\phi \left\{ -\ddot{\phi} + c^2\nabla^2\phi - \left(\frac{mc^2}{\hbar}\right)^2 \phi \right\} d^3x\, dt$$

$= 0$ for any variation $\delta\phi(\vec{x},t)$.

$$\ddot{\phi} - c^2\nabla^2\phi + \left(\frac{mc^2}{\hbar}\right)^2 \phi = 0$$

the Klein-Gordon equation

We can solve the Klein-Gordon equation, in plane waves…

$$\ddot{\phi} - c^2 \nabla^2 \phi + \left(\frac{mc^2}{\hbar}\right)^2 \phi = 0$$

the Klein-Gordon equation

$$\phi(\vec{x}, t) = C\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

where

$$-\omega^2 + c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2 = 0$$

$$\omega = \pm\sqrt{c^2 k^2 + m^2 c^4/\hbar^2}$$

I.e.,

$$\hbar\omega = \pm\sqrt{(c\hbar k)^2 + (mc^2)^2}$$

Note that this is the energy ($\hbar\omega$) and momentum ($\hbar\mathbf{k}$) relation of special relativity.

(What are the negative energy solutions?)

The general solution (**Hermitian**) is

$$\phi(\vec{x}, t) = \sum_{\vec{k}} N_k \left\{ a_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} + a_{\vec{k}}^*\, e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \right\}$$

a_k † in the quantized theory

Quantization

We may *anticipate*

$$[\, a_{\mathbf{k}} , a_{\mathbf{k}'}^\dagger \,] = \delta_{\text{Kr.}} (\, \mathbf{k} , \mathbf{k}' \,)$$

$$[\, a_{\mathbf{k}} , a_{\mathbf{k}'} \,] = 0 \quad \textit{and} \quad [\, a_{\mathbf{k}}^\dagger , a_{\mathbf{k}'}^\dagger \,] = 0$$

We'll derive this from Dirac's *canonical quantization*. Recall,

$$[\, q , p \,] = i\,\hbar \qquad \text{where} \qquad p = \partial L/\partial \dot{q}$$

$$\Pi(\vec{x}) = \frac{\delta L}{\delta \dot{\phi}(\vec{x})} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})} = \dot{\phi}(\vec{x})$$

The E.T.C.R. should be

$$[\phi(\vec{x}, t), \Pi(\vec{x}', t)] = i\hbar\, \delta^3(\vec{x} - \vec{x}')$$

Now

$$\phi(\vec{x} t) = \sum_{\vec{k}} N \{ a_{\vec{k}} e^{i(\vec{k}\cdot\vec{x} - \omega t)} + a_{\vec{k}}^{+} e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \}$$

$$\Pi(\vec{x}, t) = \sum_{\vec{k}} N (-i\omega) \{ a_{\vec{k}} e^{i(\vec{k}\cdot\vec{x} - \omega t)} - a_{\vec{k}}^{+} e^{-i(\vec{k}\cdot\vec{x} - \omega t)} \}$$

$$[\phi(\vec{x}, t), \Pi(\vec{x}', t)] = \sum_{k}\sum_{k'} N N' (-i\omega') \left[\left(a_k e^{-ik\cdot x} + a_k^{+} e^{+ik\cdot x} \right), \left(a_{k'} e^{-ik'\cdot x'} - a_{k'}^{+} e^{ik'\cdot x'} \right) \right]$$

$$k\cdot x = k^{\mu} x_{\mu} = \omega t - \vec{k}\cdot\vec{x}$$

$$= \sum_{k}\sum_{k'} N N' (-i\omega') \{ -\delta(k, k') e^{-ik\cdot(x - x')} - \delta(k, k') e^{ik\cdot(x - x')} \}$$

$$= \sum_{k} N^2 i\omega \{ e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} + e^{-i\vec{k}\cdot(\vec{x} - \vec{x}')} \}$$

= should be $i\hbar\, \delta^3(\vec{x} - \vec{x}') = i\hbar \int e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} \frac{d^3k}{(2\pi)^3}$

infinite volume limit $\sum_{\vec{k}} = \int d^3k/(2\pi)^3\, \Omega$

$= i\hbar\, \delta^3(\vec{x} - \vec{x}')$ requires that $N_k^2 = \frac{\hbar}{2\omega\Omega}$

$$N = \sqrt{\frac{\hbar}{2\omega\Omega}}$$

The Hamiltonian

$H = p\,\dot{q} - L$,

rewritten in terms of q and p

$H = \int (\pi(x)\dot{\varphi}(x) - \mathcal{L})\, d^3x$

rewritten in terms of $\varphi(x)$ and $\pi(x)$

Homework problem 18.

(A) Write H in terms of $\pi(\mathbf{x})$ and $\varphi(\mathbf{x})$.

(B) Write H in terms of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$.

Homework problem 19.

Determine the *Feynman propagator* for the free scalar field;

$\Delta_F(x - y) = \langle 0 |\, T\, \varphi(x)\, \varphi(y) \,| 0 \rangle$.

Here x stands for the 4-vector spacetime coordinate, $x^{\mu} = (x^0 , x,y,z)$.

Next: A real scalar field φ with a source ρ

- To make it simpler, set ħ = 1 and c = 1 .("natural units"). At the end of a calculation we can restore the factors of ħ and c by dimensional analysis (i.e., simple units analysis).

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \rho\phi$$

Field equation $\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \nabla\left(\frac{\partial \mathcal{L}}{\partial(\nabla\phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

$$\hookrightarrow \ddot{\phi} - \nabla^2\phi + m^2\phi - \rho = 0$$

$$(\Box + m^3)\,\phi = \rho$$

The field equation is a linear inhomogeneous equation;

so $\varphi(x,t) = \varphi_{particular}(x,t) + \varphi_{homogeneous}(x,t)$.

The *particular solution* comes from the source; e.g., it could be a mean field produced by a static source; or, waves radiated by a time dependent source. The *homogeneous solution* consists of harmonic waves.

$$-\nabla^2\phi_0 + m^2\phi_0 = \rho(\vec{x})$$

We need the Green's function of $-\nabla^2 + m^2$; i.e.,

$$(-\nabla^2 + m^2)\,G(\vec{x}-\vec{y}) = \delta^3(\vec{x}-\vec{y})$$

Then

$$\phi_0(\vec{x}) = \int G(\vec{x}-\vec{y})\,\rho(\vec{y})\,d^3y$$

The Green's function of $-\nabla^2 + m^2$

$$(-\nabla^2+m^2)\, G(\vec{\xi}) = \delta^3(\vec{\xi})$$

$$G(\vec{\xi}) = \int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\vec{k}\cdot\vec{\xi}}}{k^2+m^2}$$

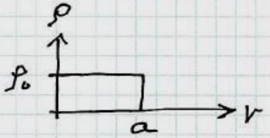
$$= \frac{e^{-m|\vec{\xi}|}}{4\pi|\vec{\xi}|}$$

(with $\hbar = 1$ and $c = 1$)

$$= \frac{e^{-mc|\vec{\xi}|/\hbar}}{4\pi|\vec{\xi}|}$$

Example

Suppose $\rho(\boldsymbol{x}) = \rho_0\, \theta(a - r)$.



$$\phi_0(\vec{x}) = \int \frac{e^{-m|\vec{x}-\vec{y}|}}{4\pi|\vec{x}-\vec{y}|}\, \rho(\vec{y})\, d^3y$$

$$\phi_0(\vec{x}) = \frac{\rho_0}{4\pi} \int e^{-m|\vec{x}-\vec{y}|}\, \frac{\theta(a-y)\, d^3y}{|\vec{x}-\vec{y}|}$$

Limiting cases —

- Large r, $\quad \phi_0(r) \sim \frac{\rho_0}{4\pi}\, \frac{e^{-mr}}{r}\, \frac{4}{3}\pi a^3$
- Small r, $\quad \phi_0(r) \sim \frac{\rho_0}{4\pi} \int e^{-my}\, \theta(a-y)\, \frac{d^3y}{y}$

$$= \frac{\rho_0}{m^2}\left\{ 1 - (1+ma)\, e^{-ma} \right\}$$

($\hbar = 1$ and $c = 1$)

An ***interaction*** Lagrangian density

$$\pounds_{\text{interaction}} = g\ \Psi^\dagger\ \Psi\ \varphi$$

- This $\pounds_{\text{int}}$ acts as a source for φ, with

  $\rho(\mathbf{x},t) = g\ \Psi^\dagger\ \Psi$.

- It also acts as a potential for Ψ :

  $V_{\text{int}}(\mathbf{x},t) = -\,g\ \varphi(\mathbf{x},t)$.

⇒ The field equations;
i.e., Lagrange's equations,

$$-\frac{\hbar^2}{2m}\nabla^2\psi - g\phi\,\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi + m^2\phi = g\,\psi^\dagger\psi$$

Homework due Fri Feb 10

Problem 18.
For the free real scalar field,
(A) Write H in terms of $\pi(\mathbf{x})$ and $\varphi(\mathbf{x})$.
(B) Write H in terms of a_k and $a_k^\dagger$.

Problem 19.
(A) Mandl and Shaw problem 3.3.
(B) Mandl and Shaw problem 3.4.

Problem 20.
The Yukawa theory problem.